

PRODUCT HARDY, BMO SPACES AND ITERATED COMMUTATORS ASSOCIATED WITH BESSEL SCHRÖDINGER OPERATORS

JORGE J. BETANCOR, XUAN THINH DUONG, JI LI, BRETT D. WICK, AND DONGYONG YANG

ABSTRACT. In this paper we establish the product Hardy spaces associated with the Bessel Schrödinger operator introduced by Muckenhoupt and Stein, and provide equivalent characterizations in terms of the Bessel Riesz transforms, non-tangential and radial maximal functions, and Littlewood–Paley theory, which are consistent with the classical product Hardy space theory developed by Chang and Fefferman. Moreover, in this specific setting, we also provide another characterization via the Telyakovskii transform, which further implies that the product Hardy space associated with this Bessel Schrödinger operator is isomorphic to the subspace of suitable “odd functions” in the standard Chang–Fefferman product Hardy space. Based on the characterizations of these product Hardy spaces, we study the boundedness of the iterated commutator of the Bessel Riesz transforms and functions in the product BMO space associated with Bessel Schrödinger operator. We show that this iterated commutator is bounded above, but does not have a lower bound.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

There are several motivations for the research carried out in this paper. Associated to the usual Laplacian Δ on \mathbb{R}^n there are several important function spaces: the Hardy space $H^1(\mathbb{R}^n)$ and the space of functions with bounded mean oscillation $BMO(\mathbb{R}^n)$. For the Hardy space, one has a family of equivalent norms that can be used to study the space: via maximal functions, square functions, area functions, Littlewood–Paley g -functions, Riesz transforms, and atomic decompositions [St93]. Similarly the space $BMO(\mathbb{R}^n)$ has different ways that it can be studied: via commutators and via Riesz transforms [CRW]. It has since become clear that the role of the differential operator greatly influences the harmonic analysis questions that one can consider.

The work of Betancor et al, [BDT], studied the Hardy space theory associated to a Bessel operator introduced by Muckenhoupt and Stein [MSt], that serves as primary motivation for our paper. Let $\lambda \in \mathbb{R}_+ := (0, \infty)$ and

$$(1.1) \quad S_\lambda f(x) := -\frac{d^2}{dx^2} f(x) + \frac{\lambda^2 - \lambda}{x^2} f(x), \quad x > 0.$$

The operator S_λ in (1.1) is a positive self-adjoint operator on $L^2(\mathbb{R}_+)$ and it can be written in divergence form as

$$S_\lambda = -x^{-\lambda} D x^{2\lambda} D x^{-\lambda} =: A_\lambda^* A_\lambda,$$

where $A_\lambda := x^\lambda D x^{-\lambda}$ and $A_\lambda^* := -x^{-\lambda} D x^\lambda$ is the adjoint operator of A_λ . There has since been numerous investigations into harmonic analysis associated to this operator. See for example [BFBMT, BCFR, BCFR2, BFS, BDT, BHN, DLWY, V, YY].

The main goal of this paper is to study the product theory of harmonic analysis associated to the operator S_λ . In particular, we establish the product Hardy spaces associated with this Bessel Schrödinger operator and provide equivalent characterizations in terms of the Bessel Riesz transforms, non-tangential and radial maximal functions, and Littlewood–Paley theory, which are consistent with the classical product Hardy space theory developed by Chang and Fefferman

Date: January 19, 2017.

2010 Mathematics Subject Classification. 42B35, 42B25, 42B30, 30L99.

Key words and phrases. Bessel operator, maximal function, Littlewood–Paley theory, Riesz transform, Cauchy–Riemann type equations, product Hardy space, product BMO space.

[CF]. We then also show that the commutators are bounded if the symbol belongs to a certain BMO space associated to the operator S_λ , but conversely this BMO does not characterize the boundedness of the commutator. This last result is surprising since it is known in the classical multi-parameter setting that these iterated commutators in fact characterize the BMO of Chang and Fefferman (see [FL, LPPW]). We now state our main results more carefully.

Throughout the paper, for every interval $I \subset \mathbb{R}_+$, we denote it by $I := I(x, t) := (x - t, x + t) \cap \mathbb{R}_+$. In the product setting $\mathbb{R}_+ \times \mathbb{R}_+$, we define $\mathfrak{R}_+ := (\mathbb{R}_+ \times \mathbb{R}_+, dx_1 dx_2)$. We work with the domain $(\mathbb{R}_+ \times \mathbb{R}_+) \times (\mathbb{R}_+ \times \mathbb{R}_+)$ and its distinguished boundary $\mathbb{R}_+ \times \mathbb{R}_+$. For $x := (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$, denote by $\Gamma(x)$ the product cone $\Gamma(x) := \Gamma_1(x_1) \times \Gamma_2(x_2)$, where $\Gamma_i(x_i) := \{(y_i, t_i) \in \mathbb{R}_+ \times \mathbb{R}_+ : |x_i - y_i| < t_i\}$ for $i := 1, 2$.

We now provide several definitions of $H_{S_\lambda}^1$. These spaces all end up being the same, which is one of the main results in this paper. This requires some additional notation, but the careful reader will notice that the spaces are distinguished notationally by a subscript to remind how they are defined.

We first define the product Hardy spaces associated with the Bessel operator S_λ using the Littlewood–Paley area functions and square functions via the semigroups $\{T_t\}_{t>0}$, where $\{T_t\}_{t>0}$ can be the Poisson semigroup $\{e^{-t\sqrt{S_\lambda}}\}_{t>0}$ or the heat semigroup $\{e^{-tS_\lambda}\}_{t>0}$. We note that the definition via heat semigroup was covered in [CDLWY] in a more general setting.

Given a function f on $L^2(\mathfrak{R}_+)$, the Littlewood–Paley area function $Sf(x)$, $x := (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$, associated with the operator S_λ is defined as

$$(1.2) \quad Sf(x) := \left(\iint_{\Gamma(x)} \left| t_1 \partial_{t_1} T_{t_1} t_2 \partial_{t_2} T_{t_2} f(y_1, y_2) \right|^2 \frac{dy_1 dy_2 dt_1 dt_2}{t_1^2 t_2^2} \right)^{\frac{1}{2}}.$$

The square function $g(f)(x)$, $x := (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$, associated with the operator S_λ is defined as

$$(1.3) \quad g(f)(x) := \left(\int_0^\infty \int_0^\infty \left| t_1 \partial_{t_1} T_{t_1} t_2 \partial_{t_2} T_{t_2} f(x_1, x_2) \right|^2 \frac{dt_1 dt_2}{t_1 t_2} \right)^{\frac{1}{2}}.$$

We define the product Hardy space via the Littlewood–Paley square functions as follows.

Definition 1.1. The Hardy space $H_g^1(\mathfrak{R}_+)$ associated with S_λ is defined as the completion of

$$\{f \in L^2(\mathfrak{R}_+) : \|g(f)\|_{L^1(\mathfrak{R}_+)} < \infty\}$$

with respect to the norm $\|f\|_{H_g^1(\mathfrak{R}_+)} := \|g(f)\|_{L^1(\mathfrak{R}_+)}$, where $g(f)$ is defined by (1.3) with $T_t := e^{-t\sqrt{S_\lambda}}$ or $T_t := e^{-tS_\lambda}$.

We now define the product Hardy space via the Littlewood–Paley area functions as follows.

Definition 1.2. The Hardy space $H_S^1(\mathfrak{R}_+)$ associated with S_λ is defined as the completion of

$$\{f \in L^2(\mathfrak{R}_+) : \|Sf\|_{L^1(\mathfrak{R}_+)} < \infty\}$$

with respect to the norm $\|f\|_{H_S^1(\mathfrak{R}_+)} := \|Sf\|_{L^1(\mathfrak{R}_+)}$, where Sf is defined by (1.2) with $T_t := e^{-t\sqrt{S_\lambda}}$ or $T_t := e^{-tS_\lambda}$.

We now define another version of the Littlewood–Paley area function. Let

$$\nabla_{t_1, y_1} := (\partial_{t_1}, \partial_{y_1}), \quad \nabla_{t_2, y_2} := (\partial_{t_2}, \partial_{y_2}).$$

Then the Littlewood–Paley area function $S_u f(x)$ for $f \in L^2(\mathfrak{R}_+)$, $x := (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$ is defined as

$$(1.4) \quad S_u f(x) := \left(\iint_{\Gamma(x)} \left| \nabla_{t_1, y_1} e^{-t_1 \sqrt{S_\lambda}} \nabla_{t_2, y_2} e^{-t_2 \sqrt{S_\lambda}}(f)(y_1, y_2) \right|^2 dy_1 dy_2 dt_1 dt_2 \right)^{\frac{1}{2}}.$$

Then naturally we have the following definition of the product Hardy space via the Littlewood–Paley area function $S_u f$.

Definition 1.3. The Hardy space $H_{S_u}^1(\mathfrak{R}_+)$ is defined as the completion of

$$\{f \in L^2(\mathfrak{R}_+) : \|S_u f\|_{L^1(\mathfrak{R}_+)} < \infty\}$$

with respect to the norm $\|f\|_{H_{S_u}^1(\mathfrak{R}_+)} := \|S_u f\|_{L^1(\mathfrak{R}_+)}.$

Next we define the product non-tangential and radial maximal functions via the heat semigroup and Poisson semigroup associated to S_λ . For all $\alpha \in (0, \infty)$, $p \in [1, \infty)$, $f \in L^p(\mathfrak{R}_+)$ and $x_1, x_2 \in \mathbb{R}_+$, let

$$\begin{aligned} \mathcal{N}_h^\alpha f(x_1, x_2) &:= \sup_{\substack{|y_1 - x_1| < \alpha t_1 \\ |y_2 - x_2| < \alpha t_2}} |e^{-t_1 S_\lambda} e^{-t_2 S_\lambda} f(y_1, y_2)|, \\ \mathcal{N}_P^\alpha f(x_1, x_2) &:= \sup_{\substack{|y_1 - x_1| < \alpha t_1 \\ |y_2 - x_2| < \alpha t_2}} |e^{-t_1 \sqrt{S_\lambda}} e^{-t_2 \sqrt{S_\lambda}} f(y_1, y_2)| \end{aligned}$$

be the product non-tangential maximal functions with aperture α via the heat semigroup and Poisson semigroup associated to S_λ , respectively. Denote $\mathcal{N}_h^1 f$ by $\mathcal{N}_h f$ and $\mathcal{N}_P^1 f$ by $\mathcal{N}_P f$. Moreover, let

$$\begin{aligned} \mathcal{R}_h f(x_1, x_2) &:= \sup_{t_1 > 0, t_2 > 0} |e^{-t_1 S_\lambda} e^{-t_2 S_\lambda} f(x_1, x_2)|, \\ \mathcal{R}_P f(x_1, x_2) &:= \sup_{t_1 > 0, t_2 > 0} |e^{-t_1 \sqrt{S_\lambda}} e^{-t_2 \sqrt{S_\lambda}} f(x_1, x_2)| \end{aligned}$$

be the product radial maximal functions via the heat semigroup and Poisson semigroup associated to S_λ , respectively.

Definition 1.4. The Hardy space $H_{\mathcal{M}}^1(\mathfrak{R}_+)$ associated to the maximal function $\mathcal{M}f$ is defined as the completion of the set

$$\{f \in L^2(\mathfrak{R}_+) : \|\mathcal{M}f\|_{L^1(\mathfrak{R}_+)} < \infty\}$$

with the norm $\|f\|_{H_{\mathcal{M}}^1(\mathfrak{R}_+)} := \|\mathcal{M}f\|_{L^1(\mathfrak{R}_+)}$. Here $\mathcal{M}f$ is one of the following maximal functions: $\mathcal{N}_h f$, $\mathcal{N}_P f$, $\mathcal{R}_h f$ and $\mathcal{R}_P f$.

Next we recall the definition of the Riesz transforms associated with S_λ . Define

$$(1.5) \quad R_{S_\lambda} f = A_\lambda S_\lambda^{-1/2} f.$$

Then we consider the definition of the product Hardy space via the Bessel Riesz transforms $R_{S_\lambda, 1}(f)$ and $R_{S_\lambda, 2}(f)$ on the first and second variable, respectively.

Definition 1.5. The product Hardy space $H_{Riesz}^1(\mathfrak{R}_+)$ is defined as the completion of

$$\{f \in L^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+) : R_{S_\lambda, 1} f, R_{S_\lambda, 2} f, R_{S_\lambda, 1} R_{S_\lambda, 2} f \in L^1(\mathfrak{R}_+)\}$$

endowed with the norm

$$\|f\|_{H_{Riesz}^1(\mathfrak{R}_+)} := \|f\|_{L^1(\mathfrak{R}_+)} + \|R_{S_\lambda, 1} f\|_{L^1(\mathfrak{R}_+)} + \|R_{S_\lambda, 2} f\|_{L^1(\mathfrak{R}_+)} + \|R_{S_\lambda, 1} R_{S_\lambda, 2} f\|_{L^1(\mathfrak{R}_+)}.$$

The first main result of this paper is as follows.

Theorem 1.6. *Let $\lambda \in (1, \infty)$. The product Hardy spaces $H_g^1(\mathfrak{R}_+)$, $H_S^1(\mathfrak{R}_+)$, $H_{S_u}^1(\mathfrak{R}_+)$, $H_{\mathcal{N}_h}^1(\mathfrak{R}_+)$, $H_{\mathcal{R}_h}^1(\mathfrak{R}_+)$, $H_{\mathcal{R}_P}^1(\mathfrak{R}_+)$, $H_{\mathcal{N}_P}^1(\mathfrak{R}_+)$ and $H_{Riesz}^1(\mathfrak{R}_+)$ coincide and have equivalent norms.*

Because we have a family of equivalent norms we now choose to use $H_{S_\lambda}^1(\mathfrak{R}_+)$ to denote the product Hardy space associated to S_λ . Based on the atomic decomposition of $H_{S_\lambda}^1(\mathfrak{R}_+)$, we see that we can identify $H_{S_\lambda}^1(\mathfrak{R}_+)$ as a closed subspace of $L^1(\mathfrak{R}_+)$.

Based on the characterization of product Hardy space $H_{S_\lambda}^1(\mathfrak{R}_+)$ via Bessel Riesz transforms and the duality of $H_{S_\lambda}^1(\mathfrak{R}_+)$ with $\text{BMO}_{S_\lambda}(\mathfrak{R}_+)$, we directly have the second result as a corollary: the decomposition of $\text{BMO}_{S_\lambda}(\mathfrak{R}_+)$. For the definition of $\text{BMO}_{S_\lambda}(\mathfrak{R}_+)$, we refer to Section 7. The proof of this result is similar to the classical setting.

Corollary 1.7. *The following two statements are equivalent:*

- (i) $\varphi \in \text{BMO}_{S_\lambda}(\mathfrak{R}_+)$;
- (ii) *There exist $g_i \in L^\infty(\mathfrak{R}_+)$, $i = 1, 2, 3, 4$, such that*

$$\varphi = g_1 + R_{S_\lambda,1}(g_2) + R_{S_\lambda,2}(g_3) + R_{S_\lambda,1}R_{S_\lambda,2}(g_4).$$

The second main result of this paper is to understand the structure of the space $H_{S_\lambda}^1(\mathfrak{R}_+)$. Given a function $f \in L^1(\mathfrak{R}_+)$, we now introduce the “product odd extension” as follows

$$(1.6) \quad f_o(x_1, x_2) := \begin{cases} f(x_1, x_2), & x_1 > 0, x_2 > 0; \\ -f(-x_1, x_2), & x_1 < 0, x_2 > 0; \\ f(-x_1, -x_2), & x_1 < 0, x_2 < 0; \\ -f(x_1, -x_2), & x_1 > 0, x_2 < 0. \end{cases}$$

Note that for this odd extension, we have, for any fixed $x_2 \in \mathbb{R}$, $f_o(x_1, x_2) = -f_o(-x_1, x_2)$; and for any fixed $x_1 \in \mathbb{R}$, $f_o(x_1, x_2) = -f_o(x_1, -x_2)$.

Then we define the Hardy space $H_o^1(\mathfrak{R}_+)$ as follows:

$$H_o^1(\mathfrak{R}_+) := \{f \in L^1(\mathfrak{R}_+) : f_o \in H^1(\mathbb{R} \times \mathbb{R})\}$$

with the norm $\|f\|_{H_o^1(\mathfrak{R}_+)} = \|f_o\|_{H^1(\mathbb{R} \times \mathbb{R})}$, where $H^1(\mathbb{R} \times \mathbb{R})$ is the standard Chang–Fefferman product Hardy space (see [CF]). This leads to the second main theorem:

Theorem 1.8. *$H_{S_\lambda}^1(\mathfrak{R}_+)$ and $H_o^1(\mathfrak{R}_+)$ coincide and they have equivalent norms.*

As corollary and application of our main Theorems 1.6 and 1.8, we also have the following results. The first one is the comparison of the classical standard Hardy space $H^1(\mathfrak{R}_+)$ and our Hardy space $H_{S_\lambda}^1(\mathfrak{R}_+)$. For the definition and properties of $H^1(\mathfrak{R}_+)$, we refer to the classical Hardy space $H^1(\mathbb{R} \times \mathbb{R})$ studied by Chang–Fefferman [CF] and modify and adapt it to \mathfrak{R}_+ , in terms of Littlewood–Paley area functions and atomic decomposition. We also refer to [HLL2] if we consider \mathfrak{R}_+ as a product space of homogeneous type. We use $\text{BMO}(\mathfrak{R}_+)$ to denote the dual space of $H^1(\mathfrak{R}_+)$. For the definition of $\text{BMO}(\mathfrak{R}_+)$ we refer to Section 7.

Theorem 1.9. *The classical product Hardy space $H^1(\mathfrak{R}_+)$ is a proper subspace of $H_{S_\lambda}^1(\mathfrak{R}_+)$. As a consequence, we obtain that $\text{BMO}_{S_\lambda}(\mathfrak{R}_+)$ is a proper subspace of the classical product BMO space $\text{BMO}(\mathfrak{R}_+)$.*

We also can provide the following result regarding the product BMO space $\text{BMO}_{S_\lambda}(\mathfrak{R}_+)$ and the iterated commutator $[[b, R_{S_\lambda,1}], R_{S_\lambda,2}]$.

Theorem 1.10. *Let $b \in \text{BMO}_{S_\lambda}(\mathfrak{R}_+)$. Then we have*

$$\|[[b, R_{S_\lambda,1}], R_{S_\lambda,2}]\|_{L^2(\mathfrak{R}_+) \rightarrow L^2(\mathfrak{R}_+)} \lesssim \|b\|_{\text{BMO}_{S_\lambda}(\mathfrak{R}_+)},$$

where the implicit constant is independent of b . However, the lower bound is NOT true. In particular, there exists a locally integrable function $b_0 \notin \text{BMO}_{S_\lambda}(\mathfrak{R}_+)$ such that

$$\|[[b_0, R_{S_\lambda,1}], R_{S_\lambda,2}]\|_{L^2(\mathfrak{R}_+) \rightarrow L^2(\mathfrak{R}_+)} \leq C_{b_0} < \infty,$$

where the constant C_{b_0} is related to b_0 .

At this point we remark that this is a novel and surprising result. In the classical multi-parameter setting, it was shown by Ferguson and Lacey, [FL], and Lacey and Terwilleger, [LT], that these iterated commutators characterize the product BMO of Chang and Fefferman. See also [LPPW, LPPW2] for the case of Riesz transforms. Whereas in this case, that natural BMO is sufficient for the boundedness of these commutators, but is not necessary.

The outline of the paper is as follows. In Section 2 we study the pointwise upper bound of the heat kernel $\mathbb{W}_t^{[\lambda]}(x, y)$ and Poisson kernel $\mathbb{P}_t^{[\lambda]}(x, y)$ of the operator S_λ . We point out

that although the potential here can be negative, $\mathbb{W}_t^{[\lambda]}(x, y)$ still satisfies the standard Gaussian upper bound and $\mathbb{P}_t^{[\lambda]}(x, y)$ satisfies the standard Poisson upper bound.

In Section 3, we prove that for $f \in L^1(\mathfrak{R}_+)$,

$$(1.7) \quad \|g(f)\|_{L^1(\mathfrak{R}_+)} \approx \|S(f)\|_{L^1(\mathfrak{R}_+)}$$

and the implicit constants are independent of f . Here the Littlewood–Paley g -function and area functions can be defined using both heat semigroups and Poisson semigroups of S_λ . The main strategy here is the atomic decomposition, especially the direction that the g -function implies the atomic decomposition, where we apply the Moser type inequality of the Poisson semigroup.

In Section 4, we prove that for $f \in L^1(\mathfrak{R}_+)$,

$$(1.8) \quad \|f\|_{H_S^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{S_u}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{N_P}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{\mathcal{R}_P}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{\mathcal{R}_h}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{N_h}^1(\mathfrak{R}_+)} \\ \lesssim \|f\|_{H_S^1(\mathfrak{R}_+)}$$

and the implicit constants are independent of f . The main approach here is the use Merryfield's Lemma, atomic decomposition, and the Moser type inequality of the Poisson semigroup.

In Section 5, we prove that for $f \in L^1(\mathfrak{R}_+)$,

$$(1.9) \quad \|f\|_{H_{\mathcal{R}_P}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{Riesz}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_S^1(\mathfrak{R}_+)},$$

and the implicit constants are independent of f . These inequalities, together with the loop in (1.8), imply that our main result Theorem 1.6 holds. The main tools we use here are the Cauchy–Riemann type equations associated with S_λ and the conjugate harmonic function estimates.

For the proof of our second main result Theorem 1.8 is as follows. We introduce a product Hardy type space $H_{\mathcal{T}}^1(\mathfrak{R}_+)$ via the Telyakovskii transform on \mathbb{R}_+ , which is also called the local Hilbert transform (see [AM] and [F]), defined by

$$\mathcal{T}_{\mathbb{R}_+} f(x) = p.v. \int_{\frac{x}{2}}^{\frac{3x}{2}} \frac{f(t)}{x-t} dt.$$

The product Hardy type space $H_{\mathcal{T}}^1(\mathfrak{R}_+)$ is defined as the completion of

$$\{f \in L^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+) : \mathcal{T}_{\mathbb{R}_+,1} f, \mathcal{T}_{\mathbb{R}_+,2} f, \mathcal{T}_{\mathbb{R}_+,1} \mathcal{T}_{\mathbb{R}_+,2} f \in L^1(\mathfrak{R}_+)\}$$

with respect to the norm

$$\|f\|_{H_{\mathcal{T}}^1(\mathfrak{R}_+)} := \|f\|_{L^1(\mathfrak{R}_+)} + \|\mathcal{T}_{\mathbb{R}_+,1} f\|_{L^1(\mathfrak{R}_+)} + \|\mathcal{T}_{\mathbb{R}_+,2} f\|_{L^1(\mathfrak{R}_+)} + \|\mathcal{T}_{\mathbb{R}_+,1} \mathcal{T}_{\mathbb{R}_+,2} f\|_{L^1(\mathfrak{R}_+)},$$

where $\mathcal{T}_{\mathbb{R}_+,1}$ denotes the Telyakovskii transform on the first variable and $\mathcal{T}_{\mathbb{R}_+,2}$ the second. Then, to prove Theorem 1.8, we demonstrate that

$$\|f\|_{H_{Riesz}^1(\mathfrak{R}_+)} \approx \|f\|_{H_{\mathcal{T}}^1(\mathfrak{R}_+)} \approx \|f\|_{H_S^1(\mathfrak{R}_+)}.$$

Finally, as applications of our main Theorems 1.6 and 1.8, in Section 6 we provide the proof of Theorems 1.9 and 1.10.

Throughout the whole paper, we denote by C positive constant which is independent of the main parameters, but it may vary from line to line. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we write $f \approx g$.

2. HEAT KERNEL AND POISSON KERNEL ESTIMATES

The heat semigroup $\{\mathbb{W}_t^{[\lambda]}\}_{t>0}$ generated by $-S_\lambda$ is defined by

$$\mathbb{W}_t^{[\lambda]}(f)(x) = \int_0^\infty \mathbb{W}_t^{[\lambda]}(x, y) f(y) dy,$$

where

$$\mathbb{W}_t^{[\lambda]}(x, y) = \frac{(xy)^{\frac{1}{2}}}{2t} I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2t}\right) e^{-\frac{x^2+y^2}{4t}}, \quad t, x, y \in (0, \infty),$$

see [BDT]. Here I_ν represents the modified Bessel function of the first kind and order ν (see [Le] for the properties of I_ν).

2.1. Upper bounds of the heat kernel and Poisson kernel.

Theorem 2.1. *The kernel $\mathbb{W}_t^{[\lambda]}(x, y)$ of the heat semigroup $\{\mathbb{W}_t^{[\lambda]}\}_{t>0}$ satisfies the Gaussian estimate. Namely, there are positive constants C and c such that for $t > 0$,*

$$(Ga) \quad |\mathbb{W}_t^{[\lambda]}(x, y)| \leq \frac{C}{\sqrt{t}} e^{-\frac{|x-y|^2}{ct}}.$$

Proof. First, we note that

$$\begin{aligned} \mathbb{W}_t^{[\lambda]}(x, y) &= \frac{(xy)^{\frac{1}{2}}}{2t} I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2t}\right) e^{-\frac{x^2+y^2}{4t}} \\ &= \left(\frac{xy}{2t}\right)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2t}\right) e^{-\frac{xy}{2t}} \cdot \frac{1}{\sqrt{2t}} e^{-\frac{(x-y)^2}{4t}}. \end{aligned}$$

Then we claim that there exists a positive constant C such that for all $x, y, t > 0$,

$$(2.1) \quad \left(\frac{xy}{2t}\right)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2t}\right) e^{-\frac{xy}{2t}} \leq C.$$

This would then prove the Theorem that the heat semigroup $\{\mathbb{W}_t^{[\lambda]}\}_{t>0}$ satisfies the Gaussian estimate (Ga). To see the claim (2.1), we first recall that (see [Le])

$$(2.2) \quad C^{-1}x^\nu \leq I_\nu(x) \leq Cx^\nu, \quad 0 < x \leq 1,$$

and that $I_\nu(x)$ has the expansion

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left\{ 1 - \frac{\mu-1}{8x} + \frac{(\mu-1)(\mu-9)}{2!(8x)^2} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8x)^3} + \dots \right\}$$

with $\mu = 4\nu^2$, which gives that

$$(2.3) \quad I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} + \Psi(x), \quad \text{with } |\Psi(x)| \leq C_\nu e^x x^{-3/2} \text{ for } x > \frac{1}{4}.$$

We now proceed by case analysis.

Case 1: $0 < \frac{xy}{2t} \leq 1$. Then from (2.2) we have

$$I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2t}\right) \approx \left(\frac{xy}{2t}\right)^{\lambda-\frac{1}{2}}.$$

So we get

$$\left(\frac{xy}{2t}\right)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2t}\right) e^{-\frac{xy}{2t}} \approx \left(\frac{xy}{2t}\right)^{\frac{1}{2}} \left(\frac{xy}{2t}\right)^{\lambda-\frac{1}{2}} e^{-\frac{xy}{2t}} \leq \left(\frac{xy}{2t}\right)^\lambda e^{-\frac{xy}{2t}} \leq C.$$

Case 2: $\frac{xy}{2t} > 1$. By (2.3) it follows

$$I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2t}\right) = \frac{e^{\frac{xy}{2t}}}{\sqrt{2\pi \frac{xy}{2t}}} + \Psi\left(\frac{xy}{2t}\right).$$

So we can write

$$\begin{aligned} \left(\frac{xy}{2t}\right)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2t}\right) e^{-\frac{xy}{2t}} &= \left(\frac{xy}{2t}\right)^{\frac{1}{2}} \frac{e^{\frac{xy}{2t}}}{\sqrt{2\pi \frac{xy}{2t}}} e^{-\frac{xy}{2t}} + \left(\frac{xy}{2t}\right)^{\frac{1}{2}} \Psi\left(\frac{xy}{2t}\right) e^{-\frac{xy}{2t}} \\ &\leq \frac{1}{\sqrt{2\pi}} + C \left(\frac{xy}{2t}\right)^{\frac{1}{2}} e^{\frac{xy}{2t}} \left(\frac{xy}{2t}\right)^{-\frac{3}{2}} e^{-\frac{xy}{2t}} \leq C. \end{aligned}$$

□

As a direct consequence, we obtain the following corollary.

Corollary 2.2. *The heat kernel $\mathbb{W}_t^{[\lambda]}(x, y)$ of the heat semigroup $\{\mathbb{W}_t^{[\lambda]}\}_{t>0}$ satisfies the Davies–Gaffney estimate. Namely, there are positive constants C and c such that for all open subsets $U_1, U_2 \subset (0, \infty)$ and all $t > 0$,*

$$(DG) \quad |\langle \mathbb{W}_t^{[\lambda]} f_1, f_2 \rangle| \leq C \exp\left(-\frac{\text{dist}(U_1, U_2)^2}{ct}\right) \|f_1\|_{L^2(0, \infty)} \|f_2\|_{L^2(0, \infty)}$$

for every $f_i \in L^2(0, \infty)$ with $\text{supp } f_i \subset U_i$, $i = 1, 2$. Here $\text{dist}(U_1, U_2) := \inf_{x \in U_1, y \in U_2} |x - y|$.

We now consider the integral kernel $\mathbb{P}_t^{[\lambda]}(x, y)$ associated with the Poisson semigroup generated by $-\sqrt{S_\lambda}$.

Corollary 2.3. *There exists a positive constant C such that*

$$(2.4) \quad |\mathbb{P}_t^{[\lambda]}(x, y)| \leq C \frac{t}{t^2 + (x - y)^2} \quad \text{for all } x, y, t > 0.$$

Proof. By the principle of subordination, we have

$$(2.5) \quad \mathbb{P}_t^{[\lambda]}(x, y) = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{1}{4s}s - \frac{3}{2}\mathbb{W}_{t^2s}^{[\lambda]}(x, y)} ds.$$

Since $\mathbb{W}_t^{[\lambda]}(x, y)$ satisfies the Gaussian upper bound (Ga), it is direct that (2.4) holds. \square

2.2. Finite propagation speed. Let us recall the finite propagation speed for the wave equation and spectral multipliers (see [DLY]) and adapted to the Bessel operator S_λ . Since the heat kernel $\mathbb{W}_t^{[\lambda]}(x, y)$ satisfies the Gaussian bound (Ga), it follows from [CS, Theorem 3] that there exists a finite, positive constant c_0 with the property that the Schwartz kernel $K_{\cos(t\sqrt{S_\lambda})}$ of $\cos(t\sqrt{S_\lambda})$ satisfies

$$(2.6) \quad \text{supp } K_{\cos(t\sqrt{S_\lambda})} \subseteq \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : |x - y| \leq c_0 t\};$$

see also [Si]. By the Fourier inversion formula, whenever F is an even, bounded, Borel function with its Fourier transform $\hat{F} \in L^1(\mathbb{R})$, we can write $F(\sqrt{S_\lambda})$ in terms of $\cos(t\sqrt{S_\lambda})$. More specifically, we have

$$(2.7) \quad F(\sqrt{S_\lambda}) = (2\pi)^{-1} \int_{-\infty}^\infty \hat{F}(t) \cos(t\sqrt{S_\lambda}) dt,$$

which, when combined with (2.6), gives

$$(2.8) \quad K_{F(\sqrt{S_\lambda})}(x, y) = (2\pi)^{-1} \int_{|t| \geq c_0^{-1}|x-y|} \hat{F}(t) K_{\cos(t\sqrt{S_\lambda})}(x, y) dt, \quad \text{for every } x, y \in \mathbb{R}_+.$$

The following two results are useful for certain estimates later. We refer to [HLMMY, Lemma 3.5] for the

Lemma 2.4. *Let $\varphi \in C_c^\infty(\mathbb{R})$ be even, $\text{supp } \varphi \subset (-c_0^{-1}, c_0^{-1})$, where c_0 is the constant in (2.6). Let Φ denote the Fourier transform of φ . Then for every $\kappa = 0, 1, 2, \dots$, and for every $t > 0$, the kernel $K_{(t^2 S_\lambda)^\kappa \Phi(t\sqrt{S_\lambda})}(x, y)$ of the operator $(t^2 S_\lambda)^\kappa \Phi(t\sqrt{S_\lambda})$ which was defined by the spectral theory, satisfies*

$$\text{supp } K_{(t^2 S_\lambda)^\kappa \Phi(t\sqrt{S_\lambda})}(x, y) \subseteq \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : |x - y| \leq t\}.$$

Lemma 2.5. *Let $\varphi \in C_c^\infty(\mathbb{R})$ be an even function with $\int_{\mathbb{R}} \varphi = 2\pi$, $\text{supp } \varphi \subset (-1, 1)$. For every $m = 0, 1, 2, \dots$, set $\Phi(\xi) := \hat{\varphi}(\xi)$, $\Phi^{(m)}(\xi) := \frac{d^m}{d\xi^m} \Phi(\xi)$. Let $\kappa, m \in \mathbb{N}$ and $\kappa + m \in 2\mathbb{N}$. Then for any $t > 0$, the kernel $K_{(t\sqrt{S_\lambda})^\kappa \Phi^{(m)}(t\sqrt{S_\lambda})}(x, y)$ of $(t\sqrt{S_\lambda})^\kappa \Phi^{(m)}(t\sqrt{L})$ satisfies*

$$(2.9) \quad \text{supp } K_{(t\sqrt{S_\lambda})^\kappa \Phi^{(m)}(t\sqrt{S_\lambda})}(x, y) \subseteq \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : |x - y| \leq t\}$$

and

$$(2.10) \quad |K_{(t\sqrt{S_\lambda})^{\kappa}\Phi^{(m)}(t\sqrt{S_\lambda})}(x, y)| \leq C t^{-1}$$

for any $x, y \in \mathbb{R}_+$.

2.3. Moser type inequality. As the Moser type inequality, established in [BCFR2, p. 454], we mean the following: For any $x_0 \in \mathbb{R}_+$, $t_0 \in (0, \infty)$ and $0 < r < t_0$,

$$(2.11) \quad |u(t_0, x_0)| \lesssim \left[\frac{1}{r^2} \int_{B((t_0, x_0), r)} |u(t, x)|^p dx dt \right]^{1/p},$$

where $u(t, x) = \mathbb{P}_t^{[\lambda]}(f)(x)$ with $f \in L^1(\mathbb{R}_+)$ and $B((t_0, x_0), r) := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ : |(t, x) - (t_0, x_0)| < r\}$. We mention that the inequality was proved in [BCFR2] for $p = 2$. However, an iteration argument shows that it also holds for general $p > 0$.

2.4. Kernel estimates of Riesz transform R_{S_λ} . We next note that the Riesz transform R_{S_λ} related to Bessel operator S_λ is bounded on $L^2(\mathbb{R}_+)$, and the kernel $R_{S_\lambda}(x, y)$ of R_{S_λ} satisfies the following size and regularity properties, as proved in [BFBMT, Proposition 4.1]:

There exists $C > 0$ such that for every $x, y \in \mathbb{R}_+$ with $x \neq y$,

$$(i) \quad |R_{S_\lambda}(x, y)| \leq \frac{C}{|x - y|};$$

$$(ii) \quad \left| \frac{\partial}{\partial x} R_{S_\lambda}(x, y) \right| + \left| \frac{\partial}{\partial y} R_{S_\lambda}(x, y) \right| \leq \frac{C}{|x - y|^2}.$$

We also recall the following version of upper bound for $R_{S_\lambda}(x, y)$, which will be very useful in Section 6, connecting to the Telyakovskii transforms. There exists constant $C > 0$ such that

$$(i)' \quad |R_{S_\lambda}(x, y)| \leq C \frac{y^\lambda}{x^{\lambda+1}}, \quad 0 < y < \frac{x}{2},$$

$$(ii)' \quad |R_{S_\lambda}(x, y)| \leq C \frac{x^{\lambda+1}}{y^{\lambda+2}}, \quad y > 2x,$$

$$(iii) \quad R_{S_\lambda}(x, y) = \frac{1}{\pi} \frac{1}{x - y} + O\left(\frac{1}{x} \left(1 + \log_+ \frac{\sqrt{xy}}{|x - y|}\right)\right), \quad \frac{x}{2} < y < 2x.$$

3. PROOF OF EQUATION (1.7)

3.1. Product Hardy spaces $H_S^1(\mathfrak{R}_+)$ and atoms. We first consider $H_S^1(\mathfrak{R}_+)$ as in Definition 1.2 via $T_t = e^{-tS_\lambda}$. Note that the kernel $\mathbb{W}_t^{[\lambda]}(x, y)$ of e^{-tS_λ} satisfies the Gaussian upper bound (Ga). Thus, $H_S^1(\mathfrak{R}_+)$ falls into the scope of the Hardy space theory developed in [DLY]. We recall the definition and the atomic decomposition as follows.

First we recall the dyadic intervals in \mathbb{R}_+ as $\mathcal{D}(\mathbb{R}_+) := \cup_{n \in \mathbb{Z}} \mathcal{D}_n(\mathbb{R}_+)$, where for each $n \in \mathbb{Z}$, $\mathcal{D}_n(\mathbb{R}_+) := \{(\frac{k}{2^n}, \frac{k+1}{2^n}] : k \in \mathbb{Z} \text{ and } k \geq 0\}$. For $I, J \in \mathcal{D}(\mathbb{R}_+)$, we use $R := I \times J$ to denote the dyadic rectangles in $\mathbb{R}_+ \times \mathbb{R}_+$. Then we denote all the dyadic rectangles in $\mathbb{R}_+ \times \mathbb{R}_+$ by $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}_+) = \cup_{n_1, n_2 \in \mathbb{Z}} \mathcal{D}_{n_1, n_2}(\mathbb{R}_+)$, where $\mathcal{D}_{n_1, n_2}(\mathbb{R}_+) = \{R = I \times J : I \in \mathcal{D}_{n_1}(\mathbb{R}_+), J \in \mathcal{D}_{n_2}(\mathbb{R}_+)\}$.

Suppose $\Omega \subset \mathbb{R}_+ \times \mathbb{R}_+$ is an open set of finite measure. Denote by $m(\Omega)$ the maximal dyadic subrectangles of Ω . Let $m_1(\Omega)$ denote those dyadic subrectangles $R \subseteq \Omega$, $R = I \times J$, that are maximal in the x_1 direction. In other words if $S = I' \times J \supseteq R$ is a dyadic subrectangle of Ω , then $I = I'$. Define $m_2(\Omega)$ similarly. Let $\tilde{\Omega} := \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{M}_s(\chi_\Omega)(x_1, x_2) > 1/2\}$, where \mathcal{M}_s is the strong maximal operator defined as

$$\mathcal{M}_s(f)(x_1, x_2) := \sup_{R: \text{rectangles in } \mathfrak{R}_+, (x_1, x_2) \in R} \frac{1}{|R|} \int_R |f(y)| dy.$$

For any $R = I \times J \in m_1(\Omega)$, we set $\gamma_1(R) = \gamma_1(R, \Omega) = \sup \frac{|I|}{|J|}$, where the supremum is taken over all dyadic intervals $l : I \subset l$ so that $l \times J \subset \tilde{\Omega}$. Define γ_2 similarly. Then Journé's lemma, (in one of its forms, see for example [J2, P, HLLin]) says, for any $\delta > 0$,

$$\sum_{R \in m_2(\Omega)} |R| \gamma_1^{-\delta}(R) \leq c_\delta |\Omega| \quad \text{and} \quad \sum_{R \in m_1(\Omega)} |R| \gamma_2^{-\delta}(R) \leq c_\delta |\Omega|$$

for some c_δ depending only on δ , not on Ω .

We now recall the definition of a $(S_\lambda, 2, M)$ -atom.

Definition 3.1 ([DLY]). Let M be a positive integer. A function $a(x_1, x_2) \in L^2(\mathfrak{R}_+)$ is called a $(S_\lambda, 2, M)$ -atom if it satisfies:

- 1) $\text{supp } a \subset \Omega$, where Ω is an open set of $\mathbb{R}_+ \times \mathbb{R}_+$ with finite measure;
- 2) a can be further decomposed into $a = \sum_{R \in m(\Omega)} a_R$ where $m(\Omega)$ is the set of all maximal dyadic

subrectangles of Ω , and there exist a series of functions b_R belonging to the domain of $S_\lambda^{k_1} \otimes S_\lambda^{k_2}$ in $L^2(\mathfrak{R}_+)$, for each $k_1, k_2 = 1, \dots, M$, such that

- (i) $a_R = (S_\lambda^M \otimes S_\lambda^M) b_R$;
- (ii) $\text{supp } (S_\lambda^{k_1} \otimes S_\lambda^{k_2}) b_R \subset 10R$, $k_1, k_2 = 0, 1, \dots, M$;
- (iii) $\|a\|_{L^2(\mathfrak{R}_+)} \leq |\Omega|^{-\frac{1}{2}}$ and

$$\sum_{R=I_R \times J_R \in m(\Omega)} \ell(I_R)^{-4M} \ell(J_R)^{-4M} \left\| (\ell(I_R)^2 S_\lambda)^{k_1} \otimes (\ell(J_R)^2 S_\lambda)^{k_2} b_R \right\|_{L^2(\mathfrak{R}_+)}^2 \leq |\Omega|^{-1}.$$

We are now able to define an atomic Hardy space $H_{at,M}^1(\mathfrak{R}_+)$ space for $M > 0$, which is equivalent to the space $H_S^1(\mathfrak{R}_+)$.

Definition 3.2 ([DLY]). Let $M > 0$. The Hardy spaces $H_{at,M}^1(\mathfrak{R}_+)$ is defined as follows. We say that $f = \sum \lambda_j a_j$ is an atomic $(S_\lambda, 2, M)$ -representation of f if $\{\lambda_j\}_{j=0}^\infty \in \ell^1$, each a_j is a $(S_\lambda, 2, M)$ -atom, and the sum converges in $L^2(\mathfrak{R}_+)$. Set

$$\mathbb{H}_{at,M}^1(\mathfrak{R}_+) := \{f \in L^2(\mathfrak{R}_+) : f \text{ has an atomic } (S_\lambda, 2, M) - \text{representation}\},$$

with the norm given by

$$(3.1) \quad \|f\|_{\mathbb{H}_{at,M}^1(\mathfrak{R}_+)} := \inf \sum_{j=0}^\infty |\lambda_j|,$$

where the infimum is taken over sequences $\{\lambda_j\}_{j=0}^\infty$ such that $\sum_{j=0}^\infty |\lambda_j| < \infty$ and $\sum_j \lambda_j a_j$ is an atomic $(S_\lambda, 2, M)$ -representation of f . The space $H_{at,M}^1(\mathfrak{R}_+)$ is then defined as the completion of $\mathbb{H}_{at,M}^1(\mathfrak{R}_+)$ with respect to this norm.

Theorem 3.3 ([DLY]). Suppose that $H_S^1(\mathfrak{R}_+)$ is as in Definition 1.2 via $T_t = e^{-tS_\lambda}$ and that $M > 0$. Then $H_S^1(\mathfrak{R}_+) = H_{at,M}^1(\mathfrak{R}_+)$. Moreover, $\|f\|_{H_S^1(\mathfrak{R}_+)} \approx \|f\|_{H_{at,M}^1(\mathfrak{R}_+)}$, where the implicit constants depend only on M .

Second, we consider $H_S^1(\mathfrak{R}_+)$ as in Definition 1.2 via $T_t = e^{-t\sqrt{S_\lambda}}$. Note that the kernel $\mathbb{P}_t^{[\lambda]}(x, y)$ of $e^{-t\sqrt{S_\lambda}}$ satisfies the Poisson upper bound. In fact, following the same approach and techniques in the proof of [DLY, Theorem 3.4], we also obtain the following result in terms of the Poisson semigroup.

Theorem 3.4. Suppose that $H_S^1(\mathfrak{R}_+)$ is as in Definition 1.2 via $T_t = e^{-t\sqrt{S_\lambda}}$ and that $M > 0$. Then $H_S^1(\mathfrak{R}_+) = H_{at,M}^1(\mathfrak{R}_+)$. Moreover, $\|f\|_{H_S^1(\mathfrak{R}_+)} \approx \|f\|_{H_{at,M}^1(\mathfrak{R}_+)}$, where the implicit constants depend only on M .

Based on the atomic decomposition above, we now show that the Hardy spaces $H_g^1(\mathfrak{R}_+)$ and $H_S^1(\mathfrak{R}_+)$ coincide and they have equivalent norms.

Theorem 3.5. *Suppose $H_g^1(\mathfrak{R}_+)$ is as in Definition 1.1 via $T_t = e^{-t\sqrt{S_\lambda}}$. Then $H_g^1(\mathfrak{R}_+) = H_S^1(\mathfrak{R}_+)$. Moreover, $\|f\|_{H_S^1(\mathfrak{R}_+)} \approx \|f\|_{H_g^1(\mathfrak{R}_+)}$ where the implicit constants are independent of f . Similar result holds for the Hardy space $H_g^1(\mathfrak{R}_+)$ as in Definition 1.1 via $T_t = e^{-t^2 S_\lambda}$.*

Proof. Consider the Hardy space $H_g^1(\mathfrak{R}_+)$ as in Definition 1.1 via $T_t = e^{-t\sqrt{S_\lambda}}$. We first show that

$$H_S^1(\mathfrak{R}_+) \subset H_g^1(\mathfrak{R}_+).$$

Let $f \in H_S^1(\mathfrak{R}_+)$. According to Theorem 3.4, $H_S^1(\mathfrak{R}_+) = H_{at,M}^1$ for $M > 0$. Then, $f \in H_{at,M}^1$, and to see that $f \in H_g^1(\mathfrak{R}_+)$, it suffices to prove that for every $(S_\lambda, 2, M)$ atom a ,

$$(3.2) \quad \|g(a)\|_{L^1(\mathfrak{R}_+)} \lesssim 1.$$

Following the same proof as in [DLY, Lemma 3.6], we obtain that the above estimate holds for the Littlewood–Paley g -function defined via Poisson or heat semigroups as in Definition 1.3.

Conversely, we show that

$$H_g^1(\mathfrak{R}_+) \subset H_S^1(\mathfrak{R}_+).$$

To see this, we will show that we can derive atomic decomposition from the Littlewood–Paley g -function defined via Poisson or heat semigroups as in Definition 1.3.

Let $f \in H_g^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$. We will first obtain the frame decomposition for f . By using the reproducing formula, setting $\psi(x) := x^{M+1}\Phi(x)$ where Φ is defined as in Lemma 2.4, we can write

$$(3.3) \quad \begin{aligned} f(x_1, x_2) &= \int_0^\infty \int_0^\infty \psi(t_1 \sqrt{S_\lambda}) \psi(t_2 \sqrt{S_\lambda}) (t_1 \sqrt{S_\lambda} e^{-t_1 \sqrt{S_\lambda}} \otimes t_2 \sqrt{S_\lambda} e^{-t_2 \sqrt{S_\lambda}})(f)(x_1, x_2) \frac{dt_1 dt_2}{t_1 t_2} \\ &= \int_0^\infty \int_0^\infty \int_{\mathfrak{R}_+} K_{\psi(t_1 \sqrt{S_\lambda})}(x_1, y_1) K_{\psi(t_2 \sqrt{S_\lambda})}(x_2, y_2) \\ &\quad (t_1 \sqrt{S_\lambda} e^{-t_1 \sqrt{S_\lambda}} \otimes t_2 \sqrt{S_\lambda} e^{-t_2 \sqrt{S_\lambda}})(f)(y_1, y_2) dy_1 dy_2 \frac{dt_1 dt_2}{t_1 t_2} \\ &= \sum_{R \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}_+)} \int_{T(R)} K_{\psi(t_1 \sqrt{S_\lambda})}(x_1, y_1) K_{\psi(t_2 \sqrt{S_\lambda})}(x_2, y_2) \\ &\quad (t_1 \sqrt{S_\lambda} e^{-t_1 \sqrt{S_\lambda}} \otimes t_2 \sqrt{S_\lambda} e^{-t_2 \sqrt{S_\lambda}})(f)(y_1, y_2) dy_1 dy_2 \frac{dt_1 dt_2}{t_1 t_2} \\ &=: \sum_{R \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}_+)} s_R W_R, \end{aligned}$$

where

$$s_R := \sup_{(t_1, y_1, t_2, y_2) \in T(R)} |R|^{1/2} |(t_1 \sqrt{S_\lambda} e^{-t_1 \sqrt{S_\lambda}} \otimes t_2 \sqrt{S_\lambda} e^{-t_2 \sqrt{S_\lambda}})(f)(y_1, y_2)|$$

and when $s_R \neq 0$,

$$W_R := \frac{1}{s_R} \int_{T(R)} K_{\psi(t_1 \sqrt{S_\lambda})}(x_1, y_1) K_{\psi(t_2 \sqrt{S_\lambda})}(x_2, y_2) (t_1 \sqrt{S_\lambda} e^{-t_1 \sqrt{S_\lambda}} \otimes t_2 \sqrt{S_\lambda} e^{-t_2 \sqrt{S_\lambda}})(f)(y_1, y_2) dy_1 dy_2 \frac{dt_1 dt_2}{t_1 t_2}.$$

Here $T(R) = I \times [\frac{|I|}{2}, |I|) \times J \times [\frac{|J|}{2}, |J|)$, $\{W_R\}_{R \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}_+)}$ is a family of frames, which satisfies the following conditions:

(1) we can further write $W_R = \sqrt{S_\lambda}^M \otimes \sqrt{S_\lambda}^M(w_R)$, where

$$w_R := \frac{1}{s_R} \int_{T(R)} K_{t_1^{M+1} \sqrt{S_\lambda} \phi(t_1 \sqrt{S_\lambda})}(x_1, y_1) K_{t_2^{M+1} \sqrt{S_\lambda} \phi(t_2 \sqrt{S_\lambda})}(x_2, y_2) \\ (t_1 \sqrt{S_\lambda} e^{-t_1 \sqrt{S_\lambda}} \otimes t_2 \sqrt{S_\lambda} e^{-t_2 \sqrt{S_\lambda}})(f)(y_1, y_2) dy_1 dy_2 \frac{dt_1 dt_2}{t_1 t_2};$$

(2) $\text{supp} \sqrt{S_\lambda}^{k_1} \otimes \sqrt{S_\lambda}^{k_2}(b_R) \subset 3R$, $k_1, k_2 = 0, 1, \dots, 2M$ and

(3) $|(\ell(I) \sqrt{S_\lambda})^{k_1} \otimes (\ell(J) \sqrt{S_\lambda})^{k_2}(b_R)| \leq \ell(I)^{2M} \ell(J)^{2M} |R|^{-1/2}$, $k_1, k_2 = 0, 1, \dots, 2M$.

For the coefficients $\{s_R\}_R$, we claim that

$$(3.4) \quad \left\| \left(\sum_{k_1, k_2 \in \mathbb{Z}} \left[\sum_{R \in \mathcal{D}_{k_1, k_2}(\mathbb{R}_+ \times \mathbb{R}_+)} |R|^{-1/2} |s_R| \chi_R(\cdot, \cdot) \right]^2 \right)^{1/2} \right\|_{L^1(\mathfrak{R}_+)} \lesssim \|g(f)\|_{L^1(\mathfrak{R}_+)}.$$

To see this, we first consider the estimate of s_R for each $R \in \mathcal{D}_{k_1, k_2}(\mathbb{R}_+ \times \mathbb{R}_+)$ with $k_1, k_2 \in \mathbb{Z}$. There exists $K, K_1, K_2 \in \mathbb{N}$ such that for every $k_1, k_2 \in \mathbb{Z}$ and $R \in \mathcal{D}_{k_1, k_2}(\mathbb{R}_+ \times \mathbb{R}_+)$, we can find $(x_{j_1}, t_{j_1}, x_{j_2}, t_{j_2}) \in T(R)$, $0 \leq j_1 \leq K_1$, $0 \leq j_2 \leq K_2$, satisfying that

- (1) $T(R) \subset \bigcup_{j_1=0}^{K_1} \bigcup_{j_2=0}^{K_2} B_{1, j_1} \times B_{2, j_2}$, where for every $0 \leq j_i \leq K_i$, $B_{i, j_i} = B((x_{j_i}, t_{j_i}), 2^{-k_i-1})$, $i = 1, 2$.
- (2) for each B_{i, j_i} , we use $\overline{B_{i, j_i}}$ to denote the projection of B_{i, j_i} onto \mathbb{R}_+ .
- (3) for every $i = 1, 2$ and $0 \leq l \leq K_i$, $\text{card}\{j : 0 \leq j \leq K_i \text{ and } B_{i, j} \cap B_{i, l} \neq \emptyset\} \leq K$.

By geometric considerations we deduce that $\bigcup_{j_1=0}^{K_1} \bigcup_{j_2=0}^{K_2} B_{1, j_1} \times B_{2, j_2} \subseteq \{(t_1, y_1, t_2, y_2) : (y_1, y_2) \in 3R, 2^{-k_1-2} < t_1 < 3 \cdot 2^{-k_1+1}, 2^{-k_2-2} < t_2 < 3 \cdot 2^{-k_2+1}\}$. Hence,

$$|s_R \chi_R(x_1, x_2)| |R|^{-1/2} \\ = \sup_{(t_1, y_1, t_2, y_2) \in T(R)} |(t_1 \sqrt{S_\lambda} e^{-t_1 \sqrt{S_\lambda}} \otimes t_2 \sqrt{S_\lambda} e^{-t_2 \sqrt{S_\lambda}})(f)(y_1, y_2)| \chi_R(x_1, x_2) \\ \leq \sum_{j_1=0}^{K_1} \sum_{j_2=0}^{K_2} \sup_{(t_1, y_1, t_2, y_2) \in B_{1, j_1} \times B_{2, j_2}} |(t_1 \sqrt{S_\lambda} e^{-t_1 \sqrt{S_\lambda}} \otimes t_2 \sqrt{S_\lambda} e^{-t_2 \sqrt{S_\lambda}})(f)(y_1, y_2)| \chi_{\overline{B_{1, j_1}} \times \overline{B_{2, j_2}}}(x_1, x_2).$$

Next, for any $q \in (0, 1)$ and for $(x_1, x_2) \in \overline{B_{1, j_1}} \times \overline{B_{2, j_2}}$,

$$\sup_{(t_1, y_1, t_2, y_2) \in B_{1, j_1} \times B_{2, j_2}} |(t_1 \sqrt{S_\lambda} e^{-t_1 \sqrt{S_\lambda}} \otimes t_2 \sqrt{S_\lambda} e^{-t_2 \sqrt{S_\lambda}})(f)(y_1, y_2)| \\ \lesssim \left(\frac{1}{|B_{1, j_1} \times B_{2, j_2}|} \int_{B_{1, j_1} \times B_{2, j_2}} |(s_1 \sqrt{S_\lambda} e^{-s_1 \sqrt{S_\lambda}} \otimes s_2 \sqrt{S_\lambda} e^{-s_2 \sqrt{S_\lambda}})(f)(y_1, y_2)|^q dy_1 dy_2 ds_1 ds_2 \right)^{\frac{1}{q}} \\ \lesssim \left[\frac{1}{|B(x_{j_1}, 2^{-k_1})| |B(x_{j_2}, 2^{-k_2})|} \int_{B(x_{j_1}, 2^{-k_1}) \times B(x_{j_2}, 2^{-k_2})} \left(\int_{2^{-k_1-2}}^{2^{-k_1+3}} \int_{2^{-k_2-2}}^{2^{-k_2+3}} |(s_1 \sqrt{S_\lambda} e^{-s_1 \sqrt{S_\lambda}} \otimes s_2 \sqrt{S_\lambda} e^{-s_2 \sqrt{S_\lambda}})(f)(y_1, y_2)|^q \frac{ds_1 ds_2}{s_1 s_2} \right) dy_1 dy_2 \right]^{\frac{1}{q}} \\ \lesssim \left[\mathcal{M}_s \left(\int_{2^{-k_1-2}}^{2^{-k_1+3}} \int_{2^{-k_2-2}}^{2^{-k_2+3}} |(s_1 \sqrt{S_\lambda} e^{-s_1 \sqrt{S_\lambda}} \otimes s_2 \sqrt{S_\lambda} e^{-s_2 \sqrt{S_\lambda}})(f)(\cdot, \cdot)|^q \frac{ds_1 ds_2}{s_1 s_2} \right) (x_1, x_2) \right]^{\frac{1}{q}},$$

where the first inequality follows from the iteration of the Moser type inequality (2.11).

Therefore, by noting that $\sum_{R \in \mathcal{D}_{k_1, k_2}(\mathbb{R}_+ \times \mathbb{R}_+)} \chi_{3R} \lesssim 1$, we have

$$\sum_{R \in \mathcal{D}_{k_1, k_2}(\mathbb{R}_+ \times \mathbb{R}_+)} \sup_{(t_1, y_1, t_2, y_2) \in T(R)} |(t_1 \sqrt{S_\lambda} e^{-t_1 \sqrt{S_\lambda}} \otimes t_2 \sqrt{S_\lambda} e^{-t_2 \sqrt{S_\lambda}})(f)(\cdot, \cdot)| \chi_R(x_1, x_2)$$

$$\begin{aligned} &\lesssim \left[\mathcal{M}_s \left(\int_{2^{-k_1-1}}^{2^{-k_1}} \int_{2^{-k_2-1}}^{2^{-k_2}} |(s_1 \sqrt{S_\lambda} e^{-s_1 \sqrt{S_\lambda}} \otimes s_2 \sqrt{S_\lambda} e^{-s_2 \sqrt{S_\lambda}})(f)(\cdot, \cdot)|^q \frac{ds_1 ds_2}{s_1 s_2} \right) (x_1, x_2) \right]^{\frac{1}{q}} \\ &=: F_{k_1, k_2}(x_1, x_2). \end{aligned}$$

We take $q \in (0, 1)$. Then, the Fefferman–Stein vector valued maximal inequality leads to

$$\begin{aligned} &\left\| \left(\sum_{k_1, k_2 \in \mathbb{Z}} \left[\sum_{R \in \mathcal{D}_{k_1, k_2}(\mathbb{R}_+ \times \mathbb{R}_+)} |R|^{-1/2} |s_R| \chi_R(\cdot, \cdot) \right]^2 \right)^{1/2} \right\|_{L^1(\mathfrak{R}_+)} \\ &\lesssim \left\| \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} F_{k_1, k_2}(x_1, x_2)^2 \right\}^{1/2} \right\|_{L^1(\mathfrak{R}_+)} = \left\| \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} F_{k_1, k_2}(x_1, x_2)^2 \right\}^{q/2} \right\|_{L^{\frac{1}{q}}(\mathfrak{R}_+)}^{\frac{1}{q}} \\ &\lesssim \left\| \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \left[\int_{2^{-k_1-1}}^{2^{-k_1}} \int_{2^{-k_2-1}}^{2^{-k_2}} |(s_1 \sqrt{S_\lambda} e^{-s_1 \sqrt{S_\lambda}} \otimes s_2 \sqrt{S_\lambda} e^{-s_2 \sqrt{S_\lambda}})(f)(x_1, x_2)|^q \frac{ds_1 ds_2}{s_1 s_2} \right]^{\frac{2}{q}} \right\}^{\frac{q}{2}} \right\|_{L^{\frac{1}{q}}(\mathfrak{R}_+)}^{\frac{1}{q}} \\ &\lesssim \left\| \left\{ \sum_{k_1, k_2 \in \mathbb{Z}} \int_{2^{-k_1-1}}^{2^{-k_1}} \int_{2^{-k_2-1}}^{2^{-k_2}} |(s_1 \sqrt{S_\lambda} e^{-s_1 \sqrt{S_\lambda}} \otimes s_2 \sqrt{S_\lambda} e^{-s_2 \sqrt{S_\lambda}})(f)(x_1, x_2)|^2 \frac{ds_1 ds_2}{s_1 s_2} \right\}^{1/2} \right\|_{L^1(\mathfrak{R}_+)} \\ &\lesssim \left\| \left\{ \int_0^\infty \int_0^\infty |(s_1 \sqrt{S_\lambda} e^{-s_1 \sqrt{S_\lambda}} \otimes s_2 \sqrt{S_\lambda} e^{-s_2 \sqrt{S_\lambda}})(f)(x_1, x_2)|^2 \frac{ds_1 ds_2}{s_1 s_2} \right\}^{1/2} \right\|_{L^1(\mathfrak{R}_+)} \\ &= \|g(f)\|_{L^1(\mathfrak{R}_+)}, \end{aligned}$$

where $g(f)$ is the Littlewood–Paley g -function defined via the Poisson semigroup $T_t = e^{-t\sqrt{S_\lambda}}$ as in Definition 1.3. This shows that the claim (3.4) holds.

Next, it suffices to show that from the frame decomposition as in (3.3),

$$f = \sum_{R \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}_+)} s_R W_R$$

with the condition (3.4) for the coefficients $\{s_R\}_{R \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}_+)}$, we can then derive the atomic decomposition.

To see this, we first denote

$$\mathcal{S}(f)(x_1, x_2) := \left(\sum_{k_1, k_2 \in \mathbb{Z}} \left[\sum_{R \in \mathcal{D}_{k_1, k_2}(\mathbb{R}_+ \times \mathbb{R}_+)} |R|^{-1/2} |s_R| \chi_R(x_1, x_2) \right]^2 \right)^{1/2}.$$

Then, we define for each $\ell \in \mathbb{Z}$,

$$\begin{aligned} \Omega_\ell &:= \left\{ (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{S}(f)(x_1, x_2) > 2^\ell \right\}, \\ B_\ell &:= \left\{ R = I_{\alpha_1}^{k_1} \times I_{\alpha_2}^{k_2} : |R \cap \Omega_\ell| > \frac{1}{2}|R|, |R \cap \Omega_{\ell+1}| \leq \frac{1}{2}|R| \right\}, \text{ and} \\ \tilde{\Omega}_\ell &:= \left\{ (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{M}_s(\chi_{\Omega_\ell})(x_1, x_2) > \frac{1}{2} \right\}, \end{aligned}$$

where \mathcal{M}_s is the strong maximal function on \mathfrak{R}_+ . Then we write

$$f = \sum_{R \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}_+)} s_R W_R = \sum_{\ell \in \mathbb{Z}} \sum_{R \in B_\ell} s_R W_R = \sum_{\ell \in \mathbb{Z}} \lambda_\ell a_\ell,$$

where $\lambda_\ell := 2^\ell |\widetilde{\Omega}_\ell|$ and

$$a_\ell := \frac{1}{\lambda_\ell} \sum_{R \in B_\ell} s_R W_R.$$

Then we can verify that these a_ℓ are the $(S_\lambda, 2, M)$ atoms as in Definition 3.1, i.e., we can obtain that $\|a_\ell\|_{L^2(\mathfrak{R}_+)} \leq C|\widetilde{\Omega}_\ell|^{-\frac{1}{2}}$, and that

$$a_\ell = \sum_{R \in m(\widetilde{\Omega}_\ell)} a_R,$$

where these a_R 's satisfy the conditions as listed in Definition 3.1.

Then we have an atomic $(S_\lambda, 2, M)$ -representation of f . The details here follow from the proof of [LS, Theorem 4.1 (ii)] and [CDLWY, Proposition 3.4]. \square

4. PROOF OF INEQUALITIES (1.8)

To this end, we will prove the chain of six inequalities as in (1.8) by the following six steps, respectively. In this section, we assume that $\lambda \geq 1$.

Step 1: $\|f\|_{H_{S_u}^1(\mathfrak{R}_+)} \leq \|f\|_{H_{S_u}^1(\mathfrak{R}_+)} \cap L^2(\mathfrak{R}_+)$ for $f \in H_{S_u}^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$.

Note that from the definitions of the area functions Sf and $S_u f$ in (1.2) and (1.4) respectively, we have for $f \in L^2(\mathfrak{R}_+)$, $S(f)(x) \leq S_u(f)(x)$, which implies that $\|f\|_{H_S^1(\mathfrak{R}_+)} \leq \|f\|_{H_{S_u}^1(\mathfrak{R}_+)}$.

Step 2: $\|f\|_{H_{S_u}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{N_P}^1(\mathfrak{R}_+)} \cap L^2(\mathfrak{R}_+)$ for $f \in H_{N_P}^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$.

We point out that the proof of $\|f\|_{H_{S_u}^1(\mathfrak{R}_+)} \leq C\|f\|_{H_{N_P}^1(\mathfrak{R}_+)}$ is similar to the proof of Step 2 in [DLWY2], assuming that we know a suitable version of the technical result originating from K. Merryfield [M]. We now build up the right version of the Merryfield type lemma in this setting. See also a similar version of Merryfield lemma in [STY] for the Schrödinger operators on \mathbb{R}^n with $n \geq 3$.

Define the gradient as

$$\nabla_{t,x} u(t, x) := (\partial_t u, \partial_x u)$$

and, as usual, the Laplace operator as

$$\Delta_{t,x} u(t, x) := \partial_t^2 u + \partial_x^2 u.$$

Lemma 4.1. *Let $\phi \in C_c^\infty(\mathbb{R}_+)$, such that $\phi \geq 0$, $\text{supp } \phi \subset (0, 1)$ and $\int_0^\infty \phi(x) dx = 1$. Then there exists a positive constant C such that for any $f, g \in L^2(\mathbb{R}_+)$ with $u(t, x) := \mathbb{P}_t^{[\lambda]} f(x)$ satisfying $\sup_{|y-x|<t} |u(t, y)| \in L^1(\mathbb{R}_+, dx)$,*

$$(4.1) \quad \begin{aligned} & \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |\nabla_{t,x} u(t, x)|^2 |\phi_t * g(x)|^2 t dx dt \\ & \leq C \left[\int_{\mathbb{R}_+} [f(x)]^2 [g(x)]^2 dx + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t, x)|^2 |Q_t(g)(x)|^2 dx dt \right], \end{aligned}$$

where $Q_t(g)(x) := (t\partial_t(\phi_t * g)(x), t\partial_x(\phi_t * g)(x), \psi_t * (g)(x))$ and $\psi(x) := x\phi(x)$, $\psi_t(x) := t^{-1}\phi(\frac{x}{t})$.

Proof. The proof of this lemma can be obtained by making minor modifications to the proof of [M, Lemma 3.1] in the classical setting, i.e., when the Laplace operator replaces the Bessel operator. For the sake of completeness and for the reader's convenience we give a brief sketch of the proof.

Note that

$$\Delta_{t,x}(u^2) = 2|\nabla_{t,x} u|^2 + 2\frac{\lambda^2 - \lambda}{x^2} u.$$

We have

$$2 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |\nabla_{t,x} u(t, x)|^2 |\phi_t * g(x)|^2 t dx dt$$

$$\begin{aligned}
&= \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \Delta_{t,x}(u^2) |\phi_t * g(x)|^2 dx dt - 2 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \frac{\lambda^2 - \lambda}{x^2} u^2(t, x) |\phi_t * g(x)|^2 dx dt \\
&\leq \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \Delta_{t,x}(u^2) |\phi_t * g(x)|^2 dx dt
\end{aligned}$$

since we have assumed the condition that $\lambda > 1$. Then integration by parts and following the proof of [M, Lemma 3.1] we get to the right-hand side of (4.1). For the construction of the function ψ , we refer to [M, Equation (3.8)]. \square

Next we have the following result for the product case. Before stating our next Lemma, we introduce the notation $\phi_{t_1} *_1 g(x_1, x_2)$, $\phi_{t_2} *_2 g(x_1, x_2)$ and $\phi_{t_1} \phi_{t_2} *_{1,2} g(x_1, x_2)$ to denote the convolution with respect to the first, second and both variables, respectively.

Lemma 4.2. *Let ϕ be a smooth function as in Lemma 4.1. There exists $C > 0$ such that for every $f, g \in L^2(\mathfrak{R}_+)$, we have that*

$$\begin{aligned}
&\iint_{\mathbb{R}_+ \times \mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |\nabla_{t_1, x_1} \nabla_{t_2, x_2} u(t_1, t_2, x_1, x_2)|^2 |\phi_{t_1} \phi_{t_2} *_{1,2} g(x_1, x_2)|^2 t_1 t_2 dx_1 dx_2 dt_1 dt_2 \\
&\leq C \left\{ \iint_{\mathbb{R}_+ \times \mathbb{R}_+} [f(x_1, x_2)]^2 [g(x_1, x_2)]^2 dx_1 dx_2 \right. \\
&\quad + \int_{\mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \mathbb{P}_{t_2}^{[\lambda]} f(x_1, x_2) \right|^2 \left[Q_{t_2}^{(2)}(g)(x_1, x_2) \right]^2 dx_2 dt_2 dx_1 \\
&\quad + \int_{\mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \mathbb{P}_{t_1}^{[\lambda]} f(x_1, x_2) \right|^2 \left[Q_{t_1}^{(1)}(g)(x_1, x_2) \right]^2 dx_1 dt_1 dx_2 \\
&\quad \left. + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t_1, t_2, x_1, x_2)|^2 \left| Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(x_1, x_2) \right|^2 \frac{dx_1 dx_2 dt_1 dt_2}{t_1 t_2} \right\},
\end{aligned}$$

where $u(t_1, t_2, x_1, x_2) := \mathbb{P}_{t_1}^{[\lambda]} \mathbb{P}_{t_2}^{[\lambda]} f(x_1, x_2)$, $Q_{t_1}^{(1)}(g)(x_1, x_2) := (t_1 \partial_{t_1} (\phi_{t_1} *_1 g)(x_1, x_2), t_1 \partial_{x_1} (\phi_{t_1} *_1 g)(x_1, x_2), \psi_{t_1} *_1 (g(\cdot, x_2))(x_1))$, and $\psi(x_1) := x_1 \phi(x_1)$, $\psi_{t_1}(x_1) := t_1^{-1} \psi(\frac{x_1}{t_1})$. The definition of $Q_{t_2}^{(2)}(g)(x_1, x_2)$ is similar.

This lemma follows from the iteration of Lemma 4.1. We omit the details.

Step 3: $\|f\|_{H_{\mathcal{N}_P}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{\mathcal{R}_P}^1(\mathfrak{R}_+)}$ for $f \in H_{\mathcal{R}_P}^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$.

Let $f \in H_{\mathcal{R}_P}^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$. We define $u(t_1, t_2, x_1, x_2) := \mathbb{P}_{t_1}^{[\lambda]} \mathbb{P}_{t_2}^{[\lambda]} f(x_1, x_2)$. For any $q \in (0, 1)$, from (2.11), for $r_1 := t_1/4$, $r_2 := t_2/4$ and for all y_1, y_2 with $|x_1 - y_1| < r_1$, $|x_2 - y_2| < r_2$, we have

$$\begin{aligned}
|u(t_1, t_2, y_1, y_2)|^q &\lesssim \frac{1}{t_1^2 t_2^2} \int_{B((x_1, t_1), r_1)} \int_{B((x_2, t_2), r_2)} |u(s_1, s_2, z_1, z_2)|^q dz_1 dz_2 ds_1 ds_2 \\
&\lesssim \frac{1}{t_1^2 t_2^2} \int_{B((x_1, t_1), r_1)} \int_{B((x_2, t_2), r_2)} \left(\sup_{s_1 > 0, s_2 > 0} |u(s_1, s_2, z_1, z_2)| \right)^q dz_1 dz_2 ds_1 ds_2 \\
&\lesssim \frac{1}{t_1 t_2} \int_{B(x_1, t_1)} \int_{B(x_2, t_2)} \mathcal{R}_P f(z_1, z_2)^q dz_1 dz_2 \\
&\lesssim \mathcal{M}_s \left((\mathcal{R}_P f)^q \right) (x_1, x_2).
\end{aligned}$$

This implies that

$$\mathcal{N}_P^{\frac{1}{4}}(f)(x_1, x_2) \lesssim \left(\mathcal{M}_s \left((\mathcal{R}_P f)^q \right) (x_1, x_2) \right)^{1/q}.$$

Note that in general

$$\|\mathcal{N}_P^a(f)\|_{L^1(\mathfrak{R}_+)} \approx \|\mathcal{N}_P^b(f)\|_{L^1(\mathfrak{R}_+)}$$

for $a, b > 0$, where the implicit constant depends only on a and b . We have

$$\|\mathcal{N}_P(f)\|_{L^1(\mathfrak{R}_+)} \lesssim \|\mathcal{N}_P^{\frac{1}{4}}(f)\|_{L^1(\mathfrak{R}_+)} \lesssim \left\| \left(\mathcal{M}_s((\mathcal{R}_P f)^q) \right)^{1/q} \right\|_{L^1(\mathfrak{R}_+)} \lesssim \|\mathcal{R}_P f\|_{L^1(\mathfrak{R}_+)}.$$

Step 4: $\|f\|_{H_{\mathcal{R}_P}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{\mathcal{R}_h}^1(\mathfrak{R}_+)}$ for $f \in H_{\mathcal{R}_h}^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$.

This inequality follows from the subordination formula (2.5). The details of the proof here are similar to the proof of Step 4 in [DLWY2].

Step 5: $\|f\|_{H_{\mathcal{R}_h}^1(\mathfrak{R}_+)} \leq \|f\|_{H_{\mathcal{N}_h}^1(\mathfrak{R}_+)}$ for $f \in H_{\mathcal{N}_h}^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$.

This inequality is clear because $\mathcal{R}_h f \leq \mathcal{N}_h f$.

Step 6: $\|f\|_{H_{\mathcal{N}_h}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{S_\lambda}^1(\mathfrak{R}_+)}$ for $f \in H_{S_\lambda}^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$.

In order to show this property it suffices to prove that for every rectangular atom a_R , as in Definition 3.1, where $R = I \times J$ is a dyadic rectangle, and $\gamma_1, \gamma_2 \geq 2$,

$$(4.2) \quad \int_{x_1 \notin \gamma_1 I} \int_0^\infty |\mathcal{N}_h(a_R)(x_1, x_2)| dx_1 dx_2 \lesssim |R|^{\frac{1}{2}} \|a_R\|_{L^2(\mathfrak{R}_+)} \gamma_1^{-1}$$

and

$$(4.3) \quad \int_0^\infty \int_{x_2 \notin \gamma_2 J} |\mathcal{N}_h(a_R)(x_1, x_2)| dx_1 dx_2 \lesssim |R|^{\frac{1}{2}} \|a_R\|_{L^2(\mathfrak{R}_+)} \gamma_2^{-1}.$$

In fact, these two inequalities follow from similar approaches and estimates from those in the proof [DLY, Lemma 3.6]. See also similar arguments in [DLWY2, Equations (4.25) and (4.26)].

5. PROOFS OF THE INEQUALITIES (1.9)

In this section, we present the proofs of inequalities (1.9) under the condition that $\lambda > 1$. Similar to [DLWY2, Section 5], the main approach here is to use the conjugate harmonic function estimates and the key tool is the Cauchy–Riemann type equations associated to S_λ . Since the techniques and concrete estimates here are quite different from those in [DLWY2], we provide the full details.

We begin with the following lemma which is a variant of [MSt, Lemma 5]. We mention that in the following lemma, we require $p \geq \lambda/(2\lambda - 1)$ which originates from a technical method from [MSt].

Lemma 5.1. *Let $\lambda \in (1, \infty)$, $p \in [\lambda/(2\lambda - 1), \infty)$ and $F := (u, v)$ with u and v satisfy the equation*

$$(5.1) \quad \begin{cases} A_\lambda u = \partial_t v, \\ \partial_t u = A_\lambda^* v. \end{cases}$$

If $|F| > 0$, then

$$(5.2) \quad \Delta|F|^p := \partial_t^2|F|^p + \partial_x^2|F|^p \geq 0.$$

Proof. We use some ideas from [MSt]. Let $\partial_t F := (\partial_t u, \partial_t v)$, $\partial_x F := (\partial_x u, \partial_x v)$, $F \cdot \partial_t F := u\partial_t u + v\partial_t v, \dots$. By (5.1), we have

$$\begin{cases} \partial_x u - \partial_t v = \frac{\lambda}{x} u, \\ \partial_t u + \partial_x v = -\frac{\lambda}{x} v, \end{cases}$$

from which we deduce that

$$\begin{cases} \partial_t^2 u + \partial_x^2 u = \frac{\lambda^2 - \lambda}{x^2} u, \\ \partial_t^2 v + \partial_x^2 v = -\frac{\lambda^2 + \lambda}{x^2} v. \end{cases}$$

Then we have

$$\begin{aligned}\Delta|F|^p &= p|F|^{p-4} \left\{ (p-2) \left[(\partial_t F \cdot F)^2 + (\partial_x F \cdot F)^2 \right] \right. \\ &\quad \left. + |F|^2 \left[\partial_t^2 F \cdot F + \partial_x^2 F \cdot F + |\partial_t F|^2 + |\partial_x F|^2 \right] \right\} \\ &= p|F|^{p-4} \left\{ (p-2) \left[(\partial_t F \cdot F)^2 + (\partial_x F \cdot F)^2 \right] \right. \\ &\quad \left. + |F|^2 \left[\frac{\lambda^2 - \lambda}{x^2} u^2 + \frac{\lambda^2 + \lambda}{x^2} v^2 + |\partial_t F|^2 + |\partial_x F|^2 \right] \right\}.\end{aligned}$$

It is obvious that $\Delta|F|^p \geq 0$ when $p \geq 2$. Thus, it remains to consider the case when $p < 2$. Then (5.2) is equivalent with

$$(5.3) \quad (\partial_t F \cdot F)^2 + (\partial_x F \cdot F)^2 \leq \frac{1}{2-p} |F|^2 \left[|\partial_t F|^2 + |\partial_x F|^2 + \frac{\lambda^2 - \lambda}{x^2} u^2 + \frac{\lambda^2 + \lambda}{x^2} v^2 \right].$$

Now let M denote the matrix

$$\begin{bmatrix} u_x & v_x \\ u_t & v_t \end{bmatrix}.$$

Then by (5.1), (5.3) can be translated to

$$(5.4) \quad |M[F]^2| \leq \frac{1}{2-p} |F|^2 \left[\|M\|^2 + \frac{(\lambda-1)(u_x - v_t)^2}{\lambda} + \frac{(\lambda+1)(u_t + v_x)^2}{\lambda} \right],$$

where $\|M\|$ denotes the ‘Hilbert-Schmidt’ norm of the matrix M .

If we consider F to be an arbitrary two-component vector, then (5.4) becomes

$$(5.5) \quad |M|^2 \leq \frac{1}{2-p} \left[\|M\|^2 + \frac{(\lambda+1)(\partial_t u + \partial_x v)^2}{\lambda} + \frac{(\lambda-1)(\partial_x u - \partial_t v)^2}{\lambda} \right],$$

where $|M|$ is the usual norm of the matrix M as an operator. Moreover, it suffices to show that

$$(5.6) \quad \max \{ \partial_x u^2, \partial_t v^2 \} \leq \frac{1}{2-p} \left[\partial_x u^2 + \partial_t v^2 + \frac{(\lambda-1)(\partial_x u - \partial_t v)^2}{\lambda} \right]$$

and

$$(5.7) \quad \max \{ \partial_t u^2, \partial_x v^2 \} \leq \frac{1}{2-p} \left[\partial_t u^2 + \partial_x v^2 + \frac{(\lambda+1)(\partial_t u + \partial_x v)^2}{\lambda} \right].$$

Arguing as in [MSt, Equation (9.10)], we see that for $\lambda > 1$, (5.6) holds if $p \geq \frac{\lambda}{2\lambda-1}$, and (5.7) holds if $p \geq \frac{\lambda}{2\lambda+1}$. Note that $\frac{\lambda}{2\lambda+1} < \frac{\lambda}{2\lambda-1}$. We then conclude that when $p \geq \frac{\lambda}{2\lambda-1}$, (5.2) holds. This finishes the proof of Lemma 5.1. \square

For $f \in L^p(\mathfrak{R}_+)$ with $p \in [1, \infty)$, and $t_1, t_2, x_1, x_2 \in \mathbb{R}_+$, let

$$(5.8) \quad u(t_1, t_2, x_1, x_2) := \mathbb{P}_{t_1}^{[\lambda]} \mathbb{P}_{t_2}^{[\lambda]} f(x_1, x_2), \quad v(t_1, t_2, x_1, x_2) := \mathbb{Q}_{t_1}^{[\lambda]} \mathbb{P}_{t_2}^{[\lambda]} f(x_1, x_2),$$

and

$$(5.9) \quad w(t_1, t_2, x_1, x_2) := \mathbb{P}_{t_1}^{[\lambda]} \mathbb{Q}_{t_2}^{[\lambda]} f(x_1, x_2), \quad z(t_1, t_2, x_1, x_2) := \mathbb{Q}_{t_1}^{[\lambda]} \mathbb{Q}_{t_2}^{[\lambda]} f(x_1, x_2),$$

where $\mathbb{Q}_{t_1}^{[\lambda]}$ and $\mathbb{Q}_{t_2}^{[\lambda]}$ are the conjugates of $\mathbb{P}_{t_1}^{[\lambda]}$ and $\mathbb{P}_{t_2}^{[\lambda]}$, respectively. For the concrete definition, we refer to (5.12) below and the following. Moreover, define

$$(5.10) \quad u^*(x_1, x_2) := \mathcal{R}_P f(x_1, x_2)$$

and

$$(5.11) \quad F(t_1, t_2, x_1, x_2) := \left\{ [u(t_1, t_2, x_1, x_2)]^2 + [v(t_1, t_2, x_1, x_2)]^2 \right. \\ \left. + [w(t_1, t_2, x_1, x_2)]^2 + [z(t_1, t_2, x_1, x_2)]^2 \right\}^{\frac{1}{2}}.$$

Next we recall the conjugate Poisson kernel and establish an auxiliary result.

Suppose $f \in L^p(\mathbb{R}_+)$, $1 \leq p < \infty$. According to [MSt, (16.5) and (16.5)'], we define, for every $x, t > 0$, the conjugate $\mathbb{Q}_t^{[\lambda]}(f)$ by

$$(5.12) \quad \mathbb{Q}_t^{[\lambda]}(f)(x) = \int_0^\infty \mathbb{Q}_t^{[\lambda]}(x, y) f(y) dy,$$

where

$$\mathbb{Q}_t^{[\lambda]}(x, y) = -(xy)^{\frac{1}{2}} \int_0^\infty e^{-tz} z J_{\lambda+\frac{1}{2}}(xz) J_{\lambda-\frac{1}{2}}(yz) dz, \quad t, x, y \in (0, \infty).$$

Lemma 5.2. *For any function $f \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ with $R_{S_\lambda} f \in L^1(\mathbb{R}_+)$,*

$$(5.13) \quad \mathbb{P}_t^{[\lambda]} f(x) = \int_0^\infty \mathbb{P}_t^{[\lambda+1]}(x, y) R_{S_\lambda} f(y) dy.$$

Proof. Indeed, let \mathcal{H}_λ denote the Hankel transform defined by

$$(5.14) \quad \mathcal{H}_\lambda f(x) := \int_0^\infty \sqrt{xy} J_{\lambda-1/2}(xy) f(y) dy, \quad x \in (0, \infty),$$

where J_ν is the Bessel function of the first kind and order ν . By using [EMOT, p.24] we can deduce that

$$(5.15) \quad \mathbb{P}_t^{[\lambda]}(x, y) = \int_0^\infty (xz)^{\frac{1}{2}} J_{\lambda-\frac{1}{2}}(xz) (yz)^{\frac{1}{2}} J_{\lambda-\frac{1}{2}}(yz) e^{-tz} dz, \quad t, x, y \in (0, \infty).$$

Also, since $f \in L^2(\mathbb{R}_+)$, according to [MSt, (16.8)], we have that

$$(5.16) \quad R_{S_\lambda}(f) = -\mathcal{H}_{\lambda+1}(\mathcal{H}_\lambda(f)).$$

By using (5.12) it follows that $\mathbb{Q}_t^{[\lambda]}(f) \in L^2(\mathbb{R}_+)$ for every $t > 0$, and

$$\begin{aligned} \mathbb{Q}_t^{[\lambda]}(f)(x) &= - \int_0^\infty f(y) \int_0^\infty e^{-tz} (xz)^{\frac{1}{2}} J_{\lambda+\frac{1}{2}}(xz) (yz)^{\frac{1}{2}} J_{\lambda-\frac{1}{2}}(yz) dz \\ &= - \int_0^\infty e^{-tz} (xz)^{\frac{1}{2}} J_{\lambda+\frac{1}{2}}(xz) \mathcal{H}_\lambda(f)(z) dz \\ &= -\mathcal{H}_{\lambda+1}(e^{-tz} \mathcal{H}_\lambda(f)(z))(x), \quad t, x \in \mathbb{R}_+. \end{aligned}$$

This interchange of integrals is justified because $f \in L^1(\mathfrak{R}_+)$ and the function $z^{\frac{1}{2}} J_\nu(z)$ is bounded on \mathbb{R}_+ when $\nu > -\frac{1}{2}$.

On the other hand, by combining (5.15) and (5.16), since $R_{S_\lambda}(f) \in L^1(\mathbb{R}_+)$, we also obtain that

$$\mathbb{P}_t^{[\lambda+1]}(R_{S_\lambda} f)(x) = -\mathcal{H}_{\lambda+1}(e^{-tz} \mathcal{H}_\lambda(f)(z))(x), \quad t, x \in \mathbb{R}_+.$$

Thus, (5.13) is proved. \square

We now establish the following lemma with respect to the harmonic conjugate functions.

Lemma 5.3. *Let $f \in H_{Riesz}^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$, F be as defined in (5.11), u, v as in (5.8) and w, z as in (5.9) Then*

$$\sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} F(t_1, t_2, x_1, x_2) dx_1 dx_2 \lesssim \|f\|_{H_{Riesz}^1(\mathfrak{R}_+)}.$$

Proof. It suffices to show that

$$(5.17) \quad \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t_1, t_2, x_1, x_2)| dx_1 dx_2 \lesssim \|f\|_{L^1(\mathfrak{R}_+)},$$

$$(5.18) \quad \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |v(t_1, t_2, x_1, x_2)| dx_1 dx_2 \lesssim \|R_{S_\lambda, 1} f\|_{L^1(\mathfrak{R}_+)},$$

$$(5.19) \quad \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |w(t_1, t_2, x_1, x_2)| dx_1 dx_2 \lesssim \|R_{S_\lambda, 2} f\|_{L^1(\mathfrak{R}_+)},$$

and

$$(5.20) \quad \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |z(t_1, t_2, x_1, x_2)| dx_1 dx_2 \lesssim \|R_{S_\lambda, 1} R_{S_\lambda, 2} f\|_{L^1(\mathfrak{R}_+)}.$$

For (5.17), note that from Corollary 2.3, $\mathbb{P}_t^{[\lambda]}(x, y)$ has the standard Poisson upper bound (i.e. (2.4)). Hence, (5.17) follows from a direct calculation by using (2.4).

Next, from (5.13) and by taking into account that the Poisson semigroup $\{\mathbb{P}_t^{[\lambda+1]}\}_{t>0}$ is uniformly bounded in $L^1(\mathbb{R}_+)$, we conclude that

$$(5.21) \quad \left\| \mathbb{Q}_t^{[\lambda]} f \right\|_{L^1(\mathbb{R}_+)} = \left\| \mathbb{P}_t^{[\lambda+1]}(R_{S_\lambda}(f)) \right\|_{L^1(\mathbb{R}_+)} \lesssim \|R_{S_\lambda} f\|_{L^1(\mathbb{R}_+)}$$

and also

$$\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |v(t_1, t_2, x_1, x_2)| dx_1 dx_2 \lesssim \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \mathbb{Q}_{t_1}^{[\lambda]} f(x_1, x_2) \right| dx_1 dx_2 \lesssim \|R_{S_\lambda, 1} f\|_{L^1(\mathfrak{R}_+)}.$$

This implies (5.18). Similarly, we have (5.19). Finally, from (5.12), we deduce that

$$z(t_1, t_2, x_1, x_2) = \int_0^\infty \int_0^\infty \mathbb{P}_{t_1}^{[\lambda+1]}(x_1, y_1) \mathbb{P}_{t_2}^{[\lambda+1]}(x_2, y_2) R_{S_\lambda, 1} R_{S_\lambda, 2} f(y_1, y_2) dy_1 dy_2,$$

which shows (5.20) immediately. This finishes the proof of Lemma 5.3. \square

Proof of Inequalities (1.9). We first show that for any $f \in H_S^1(\mathfrak{R}_+)$,

$$(5.22) \quad \|f\|_{H_{Riesz}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_S^1(\mathfrak{R}_+)}.$$

To see this, it suffices to prove that

$$\|f\|_{L^1(\mathfrak{R}_+)} + \|R_{S_\lambda, 1} f\|_{L^1(\mathfrak{R}_+)} + \|R_{S_\lambda, 2} f\|_{L^1(\mathfrak{R}_+)} + \|R_{S_\lambda, 1} R_{S_\lambda, 2} f\|_{L^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_S^1(\mathfrak{R}_+)}.$$

We point out that since the kernel $\mathbb{W}_t^{[\lambda]}(x, y)$ of the heat semigroup $\{\mathbb{W}_t^{[\lambda]}\}_{t>0}$ satisfies the Gaussian estimate (Ga) (see Theorem 2.1), The inequality $\|R_{S_\lambda, 1} R_{S_\lambda, 2} f\|_{L^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{S_\lambda}^1(\mathfrak{R}_+)}$ follows proceeding as in [DLY, Theorem 5.1]. And we consider $R_{S_\lambda, 1}$ as $R_{S_\lambda, 1} \otimes Id_2$ and $R_{S_\lambda, 2}$ as $Id_1 \otimes R_{S_\lambda, 2}$, where we use Id_1 and Id_2 to denote the identity operator on $L^2(\mathbb{R}_+)$. Then following the proof of [DLY, Theorem 5.1], we also obtain $\|R_{S_\lambda, 1} f\|_{L^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{S_\lambda}^1(\mathfrak{R}_+)}$ and $\|R_{S_\lambda, 2} f\|_{L^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{S_\lambda}^1(\mathfrak{R}_+)}$. As a consequence, we obtain that (5.22) holds.

Next, assume that $f \in H_{Riesz}^1(\mathfrak{R}_+)$. We now show that

$$(5.23) \quad \|f\|_{H_{\mathcal{R}_P}^1(\mathfrak{R}_+)} \lesssim \|f\|_{H_{Riesz}^1(\mathfrak{R}_+)}.$$

To this end, based on Lemma 5.3, it remains to prove that

$$(5.24) \quad \|f\|_{H_{\mathcal{R}_P}^1(\mathfrak{R}_+)} = \|u^*\|_{L^1(\mathfrak{R}_+)} \lesssim \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} F(t_1, t_2, x_1, x_2) dx_1 dx_2,$$

where u^* and F are as in (5.10) and (5.11). We first claim that we only need to show that for $p \in (\frac{\lambda}{2\lambda-1}, 1)$ and $\epsilon_1, t_1, \epsilon_2, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$(5.25) \quad F^p(\epsilon_1 + t_1, \epsilon_2 + t_2, x_1, x_2) \lesssim P_{t_1} P_{t_2} (F^p(\epsilon_1, \epsilon_2, \cdot, \cdot))(x_1, x_2),$$

where P_t is the classical Poisson kernel. Indeed, by Lemma 5.3, we see that $F \in L^1(\mathfrak{R}_+)$. If (5.25) holds, then the uniform $L^r(\mathbb{R}_+)$ -boundedness of $\{P_t\}_{t>0}$, with $r := 1/p$, implies that

$\{F^p(\epsilon_1, \epsilon_2, \cdot, \cdot)\}_{\epsilon_1, \epsilon_2 > 0}$ is bounded on $L^r(\mathfrak{R}_+)$. Since $L^r(\mathfrak{R}_+)$ is reflexive, there exist two sequences $\{\epsilon_{1,k}\}, \{\epsilon_{2,j}\} \downarrow 0$ and $h \in L^r(\mathfrak{R}_+)$ such that $\{F^p(\epsilon_{1,k}, \epsilon_{2,j}, \cdot, \cdot)\}_{\epsilon_{1,k}, \epsilon_{2,j} > 0}$ converges weakly to h in $L^r(\mathfrak{R}_+)$ as $k, j \rightarrow \infty$. Moreover, by Hölder's inequality, we see that

$$\begin{aligned}
 (5.26) \quad \|h\|_{L^r(\mathfrak{R}_+)}^r &= \left\{ \sup_{\|g\|_{L^{r'}(\mathfrak{R}_+)} \leq 1} \left| \iint_{\mathbb{R}_+ \times \mathbb{R}_+} g(x_1, x_2) h(x_1, x_2) dx_1 dx_2 \right| \right\}^r \\
 &= \left\{ \sup_{\|g\|_{L^{r'}(\mathfrak{R}_+)} \leq 1} \lim_{\substack{k \rightarrow \infty \\ j \rightarrow \infty}} \left| \iint_{\mathbb{R}_+ \times \mathbb{R}_+} g(x_1, x_2) F^p(\epsilon_{1,k}, \epsilon_{2,j}, \cdot, \cdot) dx_1 dx_2 \right| \right\}^r \\
 &\leq \limsup_{\substack{k \rightarrow \infty \\ j \rightarrow \infty}} \|F^p(\epsilon_{1,k}, \epsilon_{2,j}, \cdot, \cdot)\|_{L^r(\mathfrak{R}_+)}^r \\
 &\lesssim \sup_{t_1 > 0, t_2 > 0} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} F^p(t_1, t_2, x_1, x_2) dx_1 dx_2.
 \end{aligned}$$

Since F is continuous in t_1 and t_2 , for any $x_1, x_2 \in \mathbb{R}_+$,

$$F^p(t_1 + \epsilon_{1,k}, t_2 + \epsilon_{2,j}, x_1, x_2) \rightarrow F^p(t_1, t_2, x_1, x_2)$$

as $k, j \rightarrow \infty$. Observe that for each $x_1, x_2 \in (0, \infty)$,

$$P_{t_1} P_{t_2} (F^p(\epsilon_{1,k}, \epsilon_{2,j}, \cdot, \cdot))(x_1, x_2) \rightarrow P_{t_1} P_{t_2} (h)(x_1, x_2)$$

as $k, j \rightarrow \infty$. Thus, by these facts and (5.25), we have that for any $t_1, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$\begin{aligned}
 F^p(t_1, t_2, x_1, x_2) &= \lim_{\substack{k \rightarrow \infty \\ j \rightarrow \infty}} F^p(t_1 + \epsilon_{1,k}, t_2 + \epsilon_{2,j}, x_1, x_2) \\
 &\lesssim \lim_{\substack{k \rightarrow \infty \\ j \rightarrow \infty}} P_{t_1} P_{t_2} (F^p(\epsilon_{1,k}, \epsilon_{2,j}, \cdot, \cdot))(x_1, x_2) \\
 &= P_{t_1} P_{t_2} (h)(x_1, x_2).
 \end{aligned}$$

Therefore,

$$[u^*(x_1, x_2)]^p \leq \sup_{t_1 > 0, t_2 > 0} F^p(t_1, t_2, x_1, x_2) \lesssim \mathcal{M}_{R_P}(h)(x_1, x_2),$$

where \mathcal{M}_{R_P} is the classical radial maximal function. By this together with $r := 1/p$, the $L^r(\mathfrak{R}_+)$ -boundedness of \mathcal{M}_{R_P} and (5.26), we then have

$$\|u^*\|_{L^1(\mathfrak{R}_+)} \lesssim \left\| \mathcal{M}_{R_P}(h) \right\|_{L^r(\mathfrak{R}_+)}^r \lesssim \sup_{t_1 > 0, t_2 > 0} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} F(t_1, t_2, x_1, x_2) dx_1 dx_2,$$

which implies that (5.24). Thus the claim holds.

Now we prove (5.25). Observe that for any fixed $t_2, x_2 \in \mathbb{R}_+$, u, v and w, z respectively satisfy the Cauchy–Riemann equations for t_1 and x_1 , and for any fixed $t_1, x_1 \in \mathbb{R}_+$, u, w and v, z respectively satisfy the Cauchy–Riemann equations for t_2 and x_2 . That is,

$$(5.27) \quad \begin{cases} \partial_{x_1} u - \partial_{t_1} v = \frac{\lambda}{x_1} u, \\ \partial_{t_1} u + \partial_{x_1} v = -\frac{\lambda}{x_1} v; \end{cases} \quad \begin{cases} \partial_{x_1} w - \partial_{t_1} z = \frac{\lambda}{x_1} w, \\ \partial_{t_1} w + \partial_{x_1} z = -\frac{\lambda}{x_1} z; \end{cases}$$

and

$$(5.28) \quad \begin{cases} \partial_{x_2} u - \partial_{t_2} w = \frac{\lambda}{x_2} u, \\ \partial_{t_2} u + \partial_{x_2} w = -\frac{\lambda}{x_2} w; \end{cases} \quad \begin{cases} \partial_{x_2} v - \partial_{t_2} z = \frac{\lambda}{x_2} v, \\ \partial_{t_2} v + \partial_{x_2} z = -\frac{\lambda}{x_2} z. \end{cases}$$

For fixed $t_2, x_2 \in \mathbb{R}_+$, let

$$F_1(t_1, t_2, x_1, x_2) := \left\{ [u(t_1, t_2, x_1, x_2)]^2 + [v(t_1, t_2, x_1, x_2)]^2 \right\}^{\frac{1}{2}},$$

where $t_1, x_1 \in \mathbb{R}_+$. For the moment, we fix t_2, x_2 and regard F_1 as a function of t_1 and x_1 . Then we claim that:

- (1) F_1^p is subharmonic in the classical sense for $p \in (\frac{\lambda-1}{2\lambda-1}, 1]$.

Actually, this follows from (5.27), Lemma 5.1 and [SW, Theorem 4.4].

(2) for almost every $t_2 \in \mathbb{R}_+$ and almost every $x_2 \in \mathbb{R}_+$,

$$(5.29) \quad \sup_{t_1 > 0} \int_0^\infty [F_1^p(t_1, t_2, x_1, x_2)]^r dx_1 \leq \sup_{t_1 > 0} \int_0^\infty F(t_1, t_2, x_1, x_2) dx_1 < \infty.$$

To prove (5.29), we fix $t_2, x_2 \in (0, \infty)$. Then we define

$$U(t_1, x_1) := u(t_1, t_2, x_1, x_2) = \mathbb{P}_{t_1}^{[\lambda]} \mathbb{P}_{t_2}^{[\lambda]}(f)(x_1, x_2), \quad t_1, x_1 \in (0, \infty)$$

and

$$V(t_1, x_1) := v(t_1, t_2, x_1, x_2) = \mathbb{Q}_{t_1}^{[\lambda]} \mathbb{P}_{t_2}^{[\lambda]}(f)(x_1, x_2), \quad t_1, x_1 \in (0, \infty).$$

We note that

$$V(t_1, x_1) = \mathbb{P}_{t_1}^{[\lambda+1]} \mathbb{P}_{t_2}^{[\lambda]}(R_{S_{\lambda,1}}(f))(x_1, x_2), \quad t_1, x_1 \in (0, \infty).$$

Since the Poisson semigroup $\{\mathbb{P}_{t_1}^{[\lambda]}\}_{t_1 > 0}$ is uniformly bounded in $L^1(\mathbb{R}_+)$, we get

$$\sup_{t_1 > 0} \int_0^\infty |U(t_1, x_1)| dx_1 \leq C \int_0^\infty |\mathbb{P}_{t_2}^{[\lambda]}(f(x_1, \cdot))(x_2)| dx_1,$$

and

$$\sup_{t_1 > 0} \int_0^\infty |V(t_1, x_1)| dx_1 \leq C \int_0^\infty |\mathbb{P}_{t_2}^{[\lambda]}(R_{S_{\lambda,1}}(f)(x_1, \cdot))(x_2)| dx_1.$$

Then we further have

$$\begin{aligned} \sup_{t_2 > 0} \int_0^\infty \sup_{t_1 > 0} \int_0^\infty |u(t_1, t_2, x_1, x_2)| dx_1 dx_2 &\leq C \sup_{t_2 > 0} \int_0^\infty \int_0^\infty |\mathbb{P}_{t_2}^{[\lambda]}(f(x_1, \cdot))(x_2)| dx_1 dx_2 \\ &\leq C \|f\|_{L^1(\mathfrak{R}_+)} \end{aligned}$$

and

$$\begin{aligned} \sup_{t_2 > 0} \int_0^\infty \sup_{t_1 > 0} \int_0^\infty |v(t_1, t_2, x_1, x_2)| dx_1 dx_2 &\leq C \sup_{t_2 > 0} \int_0^\infty \int_0^\infty |(R_{S_{\lambda,1}}(f)(x_1, \cdot))(x_2)| dx_1 dx_2 \\ &\leq C \|R_{S_{\lambda,1}}(f)\|_{L^1(\mathfrak{R}_+)}. \end{aligned}$$

We deduce that for every $t_2 > 0$ there exists $W_{t_2} \subset \mathbb{R}_+$ such that $|W_{t_2}| = 0$ and

$$\sup_{t_1 > 0} \int_0^\infty |u(t_1, t_2, x_1, x_2)| dx_1 < \infty \quad \text{and} \quad \sup_{t_1 > 0} \int_0^\infty |v(t_1, t_2, x_1, x_2)| dx_1 < \infty$$

for every $x_2 \in \mathbb{R}_+ \setminus W_{t_2}$. Hence, there exist $W \subset \mathbb{R}_+$ with $|W| = 0$ such that

$$\sup_{t_1 > 0} \int_0^\infty |u(t_1, t_2, x_1, x_2)| dx_1 < \infty \quad \text{and} \quad \sup_{t_1 > 0} \int_0^\infty |v(t_1, t_2, x_1, x_2)| dx_1 < \infty$$

for every $x_2 \in \mathbb{R}_+ \setminus W$ and $t_2 \in \mathbb{R}_+ \cap \mathbb{Q}$, where we use \mathbb{Q} to denote the set of all rational numbers.

This shows that (5.29) holds.

From the claims (1) and (2) (for $x_2 \in \mathbb{R}_+ \setminus W$ and $t_2 \in \mathbb{R}_+ \cap \mathbb{Q}$), and from [SW, Theorem 4.6], it follows that

$$(5.30) \quad F_1^p(\epsilon_1 + t_1, t_2, x_1, x_2) \leq P_{t_1}(F_1^p(\epsilon_1, t_2, \cdot, x_2))(x_1)$$

for every $\epsilon_1, t_1, x_1 \in \mathbb{R}_+$.

Similarly, let

$$F_2(t_1, t_2, x_1, x_2) := \{[w(t_1, t_2, x_1, x_2)]^2 + [z(t_1, t_2, x_1, x_2)]^2\}^{\frac{1}{2}}.$$

By Lemma 5.1 and (5.29) with F_1 replaced by F_2 therein, [SW, Theorems 4.4 and 4.6] again, we have that for any $\epsilon_1, t_1, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$(5.31) \quad F_2^p(\epsilon_1 + t_1, t_2, x_1, x_2) \leq P_{t_1}(F_2^p(\epsilon_1, t_2, \cdot, x_2))(x_1).$$

Observe that for any $t_1, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$F(t_1, t_2, x_1, x_2) \approx \sum_{i=1}^2 F_i(t_1, t_2, x_1, x_2).$$

By this fact, (5.30) and (5.31), we have that

$$(5.32) \quad F^p(\epsilon_1 + t_1, t_2, x_1, x_2) \lesssim P_{t_1}(F^p(\epsilon_1, t_2, \cdot, \cdot))(x_1, x_2).$$

Moreover, from (5.28), Lemma 5.3, Lemma 5.1 and [SW, Theorem 4.4 and 4.6], we also deduce that

$$F^p(t_1, \epsilon_2 + t_2, x_1, x_2) \lesssim P_{t_2}(F^p(t_1, \epsilon_2, \cdot, \cdot))(x_1, x_2).$$

Now by this and (5.32), we conclude that

$$\begin{aligned} F^p(\epsilon_1 + t_1, \epsilon_2 + t_2, x_1, x_2) &\lesssim P_{t_1}(F^p(\epsilon_1, \epsilon_2 + t_2, \cdot, \cdot))(x_1, x_2) \\ &\lesssim P_{t_1}P_{t_2}(F^p(\epsilon_1, \epsilon_2, \cdot, \cdot))(x_1, x_2). \end{aligned}$$

This implies (5.25), and hence finishes the proof of (1.9). \square

6. PROOF OF SECOND MAIN RESULT: THEOREM 1.8

We recall the Telyakovskii transform, which is defined for any locally integrable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$(6.1) \quad \mathcal{T}_{\mathbb{R}_+} f(x) = p.v. \int_0^{\frac{x}{2}} \frac{f(x-t) - f(x+t)}{t} dt = p.v. \int_{\frac{x}{2}}^{\frac{3x}{2}} \frac{f(t)}{x-t} dt,$$

where the integral is defined in the Cauchy principal value sense. The operator $\mathcal{T}_{\mathbb{R}_+}$ resembles the Hilbert transform \mathcal{H} defined as

$$\mathcal{H}f(x) = p.v. \int_0^\infty \frac{f(x-t) - f(x+t)}{t} dt = p.v. \int_{-\infty}^\infty \frac{f(t)}{x-t} dt.$$

Here we omit the usual constant $1/\pi$ factor in the above definitions.

Next we consider the setting of $\mathbb{R}_+ \times \mathbb{R}_+$. We use $\mathcal{T}_{\mathbb{R}_+,1}$ to denote the Telyakovskii transform on the first variable and $\mathcal{T}_{\mathbb{R}_+,2}$ the second. Similarly for the notation of \mathcal{H}_1 and \mathcal{H}_2 . Now, as stated in the introduction, we define the product Hardy space in terms of Telyakovskii transforms.

Definition 6.1. Let $H_{\mathcal{T}}^1(\mathfrak{R}_+)$ be the completion of

$$\{f \in L^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+) : \mathcal{T}_{\mathbb{R}_+,1}f, \mathcal{T}_{\mathbb{R}_+,2}f, \mathcal{T}_{\mathbb{R}_+,1}\mathcal{T}_{\mathbb{R}_+,2}f \in L^1(\mathfrak{R}_+)\}$$

with respect to the norm

$$\|f\|_{H_{\mathcal{T}}^1(\mathfrak{R}_+)} := \|f\|_{L^1(\mathfrak{R}_+)} + \|\mathcal{T}_{\mathbb{R}_+,1}f\|_{L^1(\mathfrak{R}_+)} + \|\mathcal{T}_{\mathbb{R}_+,2}f\|_{L^1(\mathfrak{R}_+)} + \|\mathcal{T}_{\mathbb{R}_+,1}\mathcal{T}_{\mathbb{R}_+,2}f\|_{L^1(\mathfrak{R}_+)}.$$

Then we have the following structure theorem.

Theorem 6.2. $H_{\mathcal{T}}^1(\mathfrak{R}_+)$ is isomorphic to the subspace of odd functions (as defined in (1.6)) in $H^1(\mathbb{R} \times \mathbb{R})$, which is the standard Chang–Fefferman product Hardy space.

Proof. Suppose $f \in H_{\mathcal{T}}^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$. Let f_o be the product odd extension of f as defined in (1.6). We now show that $f_o \in H^1(\mathbb{R} \times \mathbb{R})$.

To see this, recalling the characterization of $H^1(\mathbb{R} \times \mathbb{R})$ via double Hilbert transforms, it suffices to show that $\mathcal{H}_1 f_o, \mathcal{H}_2 f_o, \mathcal{H}_1 \mathcal{H}_2 f_o \in L^1(\mathbb{R} \times \mathbb{R})$. Since the functions $\mathcal{H}_1 f_o, \mathcal{H}_2 f_o$ and $\mathcal{H}_1 \mathcal{H}_2 f_o$ are odd functions in terms of the first variable and second variable respectively, it suffices to show that $\mathcal{H}_1 f_o, \mathcal{H}_2 f_o, \mathcal{H}_1 \mathcal{H}_2 f_o \in L^1(\mathfrak{R}_+)$.

As for $\mathcal{H}_1 f_o$, we have by definition for every $x_1, x_2 > 0$,

$$(6.2) \quad \mathcal{H}_1 f_o(x_1, x_2) - \mathcal{T}_{\mathbb{R}_+,1}f(x_1, x_2) = 2I_1(f)(x_1, x_2) + 2I_2(f)(x_1, x_2) - I_3(f)(x_1, x_2),$$

where

$$\begin{aligned} I_1(f)(x_1, x_2) &= \int_0^{\frac{x_1}{2}} f(t_1, x_2) \frac{t_1}{x_1^2 - t_1^2} dt_1 \\ I_2(f)(x_1, x_2) &= \int_{\frac{3x_1}{2}}^{\infty} f(t_1, x_2) \frac{t_1}{x_1^2 - t_1^2} dt_1 \\ I_3(f)(x_1, x_2) &= \int_{\frac{x_1}{2}}^{\frac{3x_1}{2}} f(t_1, x_2) \frac{1}{x_1 + t_1} dt_1. \end{aligned}$$

A direct calculation shows that

$$\begin{aligned} \|I_1(f)\|_{L^1(\mathfrak{R}_+)} &\leq \int_0^\infty \int_0^\infty \int_0^{\frac{x_1}{2}} |f(t_1, x_2)| \frac{t_1}{x_1^2 - t_1^2} dt_1 dx_1 dx_2 = \ln \sqrt{3} \|f\|_{L^1(\mathfrak{R}_+)}, \\ \|I_2(f)\|_{L^1(\mathfrak{R}_+)} &\leq \int_0^\infty \int_0^\infty \int_{\frac{3x_1}{2}}^\infty |f(t_1, x_2)| \frac{t_1}{|x_1^2 - t_1^2|} dt_1 dx_1 dx_2 = \ln \sqrt{5} \|f\|_{L^1(\mathfrak{R}_+)}, \\ \|I_3(f)\|_{L^1(\mathfrak{R}_+)} &\leq \int_0^\infty \int_0^\infty \int_{\frac{x_1}{2}}^{\frac{3x_1}{2}} |f(t_1, x_2)| \frac{1}{x_1 + t_1} dt_1 dx_1 dx_2 = \ln(5/3) \|f\|_{L^1(\mathfrak{R}_+)}. \end{aligned}$$

Hence we obtain that

$$(6.3) \quad \|\mathcal{H}_1 f_o\|_{L^1(\mathfrak{R}_+)} \leq \|\mathcal{T}_{\mathbb{R}_+, 1} f\|_{L^1(\mathfrak{R}_+)} + C \|f\|_{L^1(\mathfrak{R}_+)}.$$

As for $\mathcal{H}_2 f_o$, note that by definition, for $x_1, x_2 > 0$,

$$(6.4) \quad \mathcal{H}_2 f_o(x_1, x_2) - \mathcal{T}_{\mathbb{R}_+, 2} f(x_1, x_2) = 2J_1(f)(x_1, x_2) + 2J_2(f)(x_1, x_2) - J_3(f)(x_1, x_2),$$

where

$$\begin{aligned} J_1(f)(x_1, x_2) &= \int_0^{\frac{x_2}{2}} f(x_1, t_2) \frac{t_2}{x_2^2 - t_2^2} dt_2 \\ J_2(f)(x_1, x_2) &= \int_{\frac{3x_2}{2}}^\infty f(x_1, t_2) \frac{t_2}{|x_2^2 - t_2^2|} dt_2 \\ J_3(f)(x_1, x_2) &= \int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} f(x_1, t_2) \frac{1}{x_2 + t_2} dt_2. \end{aligned}$$

Again, a direct calculation shows that

$$\begin{aligned} \|J_1(f)\|_{L^1(\mathfrak{R}_+)} &\leq \int_0^\infty \int_0^\infty \int_0^{\frac{x_2}{2}} |f(x_1, t_2)| \frac{t_2}{x_2^2 - t_2^2} dt_2 dx_2 dx_1 = \ln \sqrt{3} \|f\|_{L^1(\mathfrak{R}_+)}, \\ \|J_2(f)\|_{L^1(\mathfrak{R}_+)} &\leq \int_0^\infty \int_0^\infty \int_{\frac{3x_2}{2}}^\infty |f(x_1, t_2)| \frac{t_2}{|x_2^2 - t_2^2|} dt_2 dx_2 dx_1 = \ln \sqrt{5} \|f\|_{L^1(\mathfrak{R}_+)}, \\ \|J_3(f)\|_{L^1(\mathfrak{R}_+)} &\leq \int_0^\infty \int_0^\infty \int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} |f(x_1, t_2)| \frac{1}{x_2 + t_2} dt_2 dx_2 dx_1 = \ln(5/3) \|f\|_{L^1(\mathfrak{R}_+)}. \end{aligned}$$

And, hence we obtain that

$$(6.5) \quad \|\mathcal{H}_2 f_o\|_{L^1(\mathfrak{R}_+)} \leq \|\mathcal{T}_{\mathbb{R}_+, 2} f\|_{L^1(\mathfrak{R}_+)} + C \|f\|_{L^1(\mathfrak{R}_+)}.$$

As for $\mathcal{H}_1 \mathcal{H}_2 f_o$, note that by definition, for $x_1 > 0, x_2 > 0$,

$$\begin{aligned} (6.6) \quad &\mathcal{H}_1 \mathcal{H}_2 f_o(x_1, x_2) - \mathcal{T}_{\mathbb{R}_+, 1} \mathcal{H}_2 f_o(x_1, x_2) \\ &= 2 \int_0^{\frac{x_1}{2}} \mathcal{H}_2 f_o(t_1, x_2) \frac{t_1}{x_1^2 - t_1^2} dt_1 + 2 \int_{\frac{3x_1}{2}}^\infty \mathcal{H}_2 f_o(t_1, x_2) \frac{t_1}{x_1^2 - t_1^2} dt_1 \end{aligned}$$

$$\begin{aligned}
& - \int_{\frac{x_1}{2}}^{\frac{3x_1}{2}} \mathcal{H}_2 f_o(t_1, x_2) \frac{1}{x_1 + t_1} dt_1 \\
& = 2I_1 \left(\mathcal{H}_2 f_o \right) (x_1, x_2) + 2I_2 \left(\mathcal{H}_2 f_o \right) (x_1, x_2) - I_3 \left(\mathcal{H}_2 f_o \right) (x_1, x_2).
\end{aligned}$$

According to the estimates of I_1 , I_2 and I_3 above, we have that the $L^1(\mathfrak{R}_+)$ norm of the three terms in the right-hand side of (6.6) is bounded by $\|\mathcal{H}_2 f_o\|_{L^1(\mathfrak{R}_+)}$, which is further controlled by $\|\mathcal{T}_{\mathbb{R}_+, 2} f\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)}$ as showed in (6.5). Thus, it is easy to see that

$$\|\mathcal{H}_1 \mathcal{H}_2 f_o\|_{L^1(\mathfrak{R}_+)} \leq \|\mathcal{T}_{\mathbb{R}_+, 1} \mathcal{H}_2 f_o\|_{L^1(\mathfrak{R}_+)} + \|\mathcal{T}_{\mathbb{R}_+, 2} f\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)}.$$

Moreover, for the term $\mathcal{T}_{\mathbb{R}_+, 1} \mathcal{H}_2 f_o(x_1, x_2)$, we have

$$\begin{aligned}
(6.7) \quad \mathcal{T}_{\mathbb{R}_+, 1} \mathcal{H}_2 f_o(x_1, x_2) - \mathcal{T}_{\mathbb{R}_+, 1} \mathcal{T}_{\mathbb{R}_+, 2} f(x_1, x_2) &= 2 \int_{\frac{x_1}{2}}^{\frac{3x_1}{2}} \int_0^{\frac{x_2}{2}} f(t_1, t_2) \frac{t_2}{x_2^2 - t_2^2} \frac{1}{x_1 - t_1} dt_2 dt_1 \\
&+ 2 \int_{\frac{x_1}{2}}^{\frac{3x_1}{2}} \int_{\frac{3x_2}{2}}^{\infty} f(t_1, t_2) \frac{t_2}{x_2^2 - t_2^2} \frac{1}{x_1 - t_1} dt_2 dt_1 \\
&- \int_{\frac{x_1}{2}}^{\frac{3x_1}{2}} \int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} f(t_1, t_2) \frac{1}{x_2 + t_2} \frac{1}{x_1 - t_1} dt_2 dt_1 \\
&=: K_1 + K_2 + K_3.
\end{aligned}$$

We now consider K_1 . First note that for $f \in H^1_{\mathcal{T}}(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$,

$$K_1 = 2 \int_0^{\frac{x_2}{2}} \mathcal{T}_{\mathbb{R}_+, 1} f(x_1, t_2) \frac{t_2}{x_2^2 - t_2^2} dt_2 = 2J_1 \left(\mathcal{T}_{\mathbb{R}_+, 1} f \right) (x_1, x_2).$$

In fact, this follows from the facts that $\mathcal{T}_{\mathbb{R}_+, 1}$ is bounded on $L^2(\mathfrak{R}_+)$ (see [AM, Lemma 1]) and that J_1 is also bounded on $L^2(\mathfrak{R}_+)$, which follows from a direct calculation.

Then, by noting that $\mathcal{T}_{\mathbb{R}_+, 1} f \in L^1(\mathfrak{R}_+)$ and according to the estimates of J_1 above, we have

$$\|K_1\|_{L^1(\mathfrak{R}_+)} \leq C\|\mathcal{T}_{\mathbb{R}_+, 1} f\|_{L^1(\mathfrak{R}_+)}.$$

Again, according to the estimates of J_2 and J_3 above, we have that the $L^1(\mathfrak{R}_+)$ norms of K_2 and K_3 are both bounded by $C\|\mathcal{T}_{\mathbb{R}_+, 1} f\|_{L^1(\mathfrak{R}_+)}$. Here, the singular integrals must be understood as principal values. Thus, it is easy to see that

$$\|\mathcal{T}_{\mathbb{R}_+, 1} \mathcal{H}_2 f_o\|_{L^1(\mathfrak{R}_+)} \leq \|\mathcal{T}_{\mathbb{R}_+, 1} \mathcal{T}_{\mathbb{R}_+, 2} f\|_{L^1(\mathfrak{R}_+)} + C\|\mathcal{T}_{\mathbb{R}_+, 1} f\|_{L^1(\mathfrak{R}_+)}.$$

Combining these estimates, we have

$$\begin{aligned}
(6.8) \quad & \|\mathcal{H}_1 \mathcal{H}_2 f_o\|_{L^1(\mathfrak{R}_+)} \\
& \leq \|\mathcal{T}_{\mathbb{R}_+, 1} \mathcal{T}_{\mathbb{R}_+, 2} f\|_{L^1(\mathfrak{R}_+)} + C\|\mathcal{T}_{\mathbb{R}_+, 1} f\|_{L^1(\mathfrak{R}_+)} + C\|\mathcal{T}_{\mathbb{R}_+, 2} f\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)}.
\end{aligned}$$

Hence, combining the estimates in (6.3), (6.5) and (6.8), we obtain that $\mathcal{H}_1 f_o, \mathcal{H}_2 f_o, \mathcal{H}_1 \mathcal{H}_2 f_o \in L^1(\mathfrak{R}_+)$, which in turn gives $\mathcal{H}_1 f_o, \mathcal{H}_2 f_o, \mathcal{H}_1 \mathcal{H}_2 f_o \in L^1(\mathbb{R} \times \mathbb{R})$, i.e., $f_o \in H^1(\mathbb{R} \times \mathbb{R})$.

Conversely, based on the same estimates above, we can also obtain that for $f \in L^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$, if $f_o \in H^1(\mathbb{R} \times \mathbb{R})$ then we have the following estimates:

$$(6.9) \quad \|\mathcal{T}_{\mathbb{R}_+, 1} f\|_{L^1(\mathfrak{R}_+)} \leq \|\mathcal{H}_1 f_o\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)},$$

which follows from the equality (6.2) and the estimates for (6.3);

$$(6.10) \quad \|\mathcal{T}_{\mathbb{R}_+, 2} f\|_{L^1(\mathfrak{R}_+)} \leq \|\mathcal{H}_2 f_o\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)},$$

which follows from the equality (6.4) and the estimates for (6.5); and

$$\begin{aligned}
(6.11) \quad & \|\mathcal{T}_{\mathbb{R}_+, 1} \mathcal{T}_{\mathbb{R}_+, 2} f\|_{L^1(\mathfrak{R}_+)} \\
& \leq C(\|\mathcal{H}_1 \mathcal{H}_2 f\|_{L^1(\mathfrak{R}_+)} + \|\mathcal{H}_1 f\|_{L^1(\mathfrak{R}_+)} + \|\mathcal{H}_2 f\|_{L^1(\mathfrak{R}_+)} + \|f\|_{L^1(\mathfrak{R}_+)}),
\end{aligned}$$

which follows from the equalities (6.6) and (6.7), and from the estimates for (6.8).

Estimates (6.9), (6.10) and (6.11) combined together give that $f \in H_{\mathcal{T}}^1(\mathfrak{R}_+)$. \square

Theorem 6.3. *The Hardy spaces $H_{Riesz}^1(\mathfrak{R}_+)$ and $H_{\mathcal{T}}^1(\mathfrak{R}_+)$ coincide and they have equivalent norms.*

Proof. Now suppose $f \in H_{\mathcal{T}}^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$. We will show that f is in $H_{Riesz}^1(\mathfrak{R}_+)$, i.e., we need to verify that $R_{S_{\lambda},1}(f)$, $R_{S_{\lambda},2}(f)$ and $R_{S_{\lambda},1}R_{S_{\lambda},2}(f)$ are all in $L^1(\mathfrak{R}_+)$.

One observes that the Riesz transform $R_{S_{\lambda},1}f$ can be written as

$$(6.12) \quad R_{S_{\lambda},1}(f)(x_1, x_2) =: A_1(f)(x_1, x_2) + A_2(f)(x_1, x_2) + A_3(f)(x_1, x_2) + A_4(f)(x_1, x_2),$$

where

$$\begin{aligned} A_1(f)(x_1, x_2) &:= \int_0^{\frac{x_1}{2}} R_{S_{\lambda},1}(x_1, y_1) f(y_1, x_2) dy_1 \\ A_2(f)(x_1, x_2) &:= \left(p.v. \int_{\frac{x_1}{2}}^{\frac{3x_1}{2}} R_{S_{\lambda},1}(x_1, y_1) f(y_1, x_2) dy_1 - \frac{1}{\pi} \mathcal{T}_{\mathbb{R}_+,1}(f)(x_1, x_2) \right) \\ A_3(f)(x_1, x_2) &:= \int_{\frac{3x_1}{2}}^{\infty} R_{S_{\lambda},1}(x_1, y_1) f(y_1, x_2) dy_1 \\ A_4(f)(x_1, x_2) &:= \frac{1}{\pi} \mathcal{T}_{\mathbb{R}_+,1}(f)(x_1, x_2). \end{aligned}$$

Symmetrically, we can write

$$(6.13) \quad R_{S_{\lambda},2}(f)(x_1, x_2) =: B_1(f)(x_1, x_2) + B_2(f)(x_1, x_2) + B_3(f)(x_1, x_2) + B_4(f)(x_1, x_2),$$

where

$$\begin{aligned} B_1(f)(x_1, x_2) &:= \int_0^{\frac{x_2}{2}} R_{S_{\lambda},2}(x_2, y_2) f(x_1, y_2) dy_2 \\ B_2(f)(x_1, x_2) &:= \left(p.v. \int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} R_{S_{\lambda},2}(x_2, y_2) f(x_1, y_2) dy_2 - \frac{1}{\pi} \mathcal{T}_{\mathbb{R}_+,2}(f)(x_1, x_2) \right) \\ B_3(f)(x_1, x_2) &:= \int_{\frac{3x_2}{2}}^{\infty} R_{S_{\lambda},2}(x_2, y_2) f(x_1, y_2) dy_2 \\ B_4(f)(x_1, x_2) &:= \frac{1}{\pi} \mathcal{T}_{\mathbb{R}_+,2}(f)(x_1, x_2). \end{aligned}$$

From the kernel upper bound (i)' in Section 2.4, we obtain that

$$(6.14) \quad \|A_1(f)\|_{L^1(\mathfrak{R}_+)} + \|B_1(f)\|_{L^1(\mathfrak{R}_+)} \leq C\|f\|_{L^1(\mathfrak{R}_+)},$$

and similarly, from the kernel upper bound (ii)' in Section 2.4, we obtain that

$$(6.15) \quad \|A_3(f)\|_{L^1(\mathfrak{R}_+)} + \|B_3(f)\|_{L^1(\mathfrak{R}_+)} \leq C\|f\|_{L^1(\mathfrak{R}_+)},$$

Next, from the kernel upper bound (iii) in Section 2.4, we obtain that

$$|A_2(f)(x_1, x_2)| \leq C \int_{\frac{x_1}{2}}^{\frac{3x_1}{2}} \frac{1}{y_1} \left(1 + \log_+ \left(1 + \frac{\sqrt{x_1 y_1}}{|x_1 - y_1|} \right) \right) |f(y_1, x_2)| dy_1.$$

And from this, it is direct calculation to verify

$$(6.16) \quad \|A_2(f)\|_{L^1(\mathfrak{R}_+)} \leq C\|f\|_{L^1(\mathfrak{R}_+)}.$$

Similarly, by direction calculation

$$(6.17) \quad \|B_2(f)\|_{L^1(\mathfrak{R}_+)} \leq C\|f\|_{L^1(\mathfrak{R}_+)}.$$

We now show that the function $R_{S_\lambda,1}(f)$ is in $L^1(\mathfrak{R}_+)$. In fact, from the equality (6.12) and the estimates in (6.14), (6.15) and (6.16), we obtain that

$$(6.18) \quad \begin{aligned} \|R_{S_\lambda,1}(f)\|_{L^1(\mathfrak{R}_+)} &\leq \|A_4(f)\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)} \\ &\leq C\|\mathcal{T}_{\mathbb{R}_+,1}(f)\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)}. \end{aligned}$$

Similarly, we get that the function $R_{S_\lambda,2}(f)$ is in $L^1(\mathfrak{R}_+)$, which follows from the equality (6.13) and the estimates in (6.14), (6.15) and (6.17). Moreover, we have

$$(6.19) \quad \begin{aligned} \|R_{S_\lambda,2}(f)\|_{L^1(\mathfrak{R}_+)} &\leq \|B_4(f)\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)} \\ &\leq C\|\mathcal{T}_{\mathbb{R}_+,2}(f)\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)}. \end{aligned}$$

We now consider $R_{S_\lambda,1}R_{S_\lambda,2}(f)$. From the equalities (6.12) and (6.13), we have we obtain that

$$(6.20) \quad R_{S_\lambda,1}R_{S_\lambda,2}(f)(x_1, x_2) = \sum_{i=1}^4 \sum_{j=1}^4 A_i B_j(f)(x_1, x_2).$$

From the kernel upper bounds (i)' and (ii)' in Section 2.4, and the estimates for A_2 and B_2 above, we obtain that

$$(6.21) \quad \sum_{i=1}^3 \sum_{j=1}^3 \|A_i B_j(f)\|_{L^1(\mathfrak{R}_+)} \leq C\|f\|_{L^1(\mathfrak{R}_+)},$$

Based on the estimate in (6.21), we obtain that $R_{S_\lambda,1}R_{S_\lambda,2}(f) \in L^1(\mathfrak{R}_+)$ and we have

$$(6.22) \quad \begin{aligned} \|R_{S_\lambda,1}R_{S_\lambda,2}(f)\|_{L^1(\mathfrak{R}_+)} &\leq C \sum_{i=1}^3 \|A_i B_4(f)\|_{L^1(\mathfrak{R}_+)} + C \sum_{j=1}^3 \|A_4 B_j(f)\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)} \\ &\leq C\|\mathcal{T}_{\mathbb{R}_+,1}\mathcal{T}_{\mathbb{R}_+,2}f\|_{L^1(\mathfrak{R}_+)} + C\|\mathcal{T}_{\mathbb{R}_+,2}f\|_{L^1(\mathfrak{R}_+)} + C\|\mathcal{T}_{\mathbb{R}_+,1}f\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)}. \end{aligned}$$

Combining the estimates in (6.18), (6.19) and (6.22), we obtain that f is in $H_{Riesz}^1(\mathfrak{R}_+)$.

Conversely, suppose $f \in H_{Riesz}^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$. From the equalities (6.12), (6.13) and (6.20), we obtain that

$$\begin{aligned} \|\mathcal{T}_{\mathbb{R}_+,1}(f)\|_{L^1(\mathfrak{R}_+)} &\leq C\|R_{S_\lambda,1}(f)\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)}, \\ \|\mathcal{T}_{\mathbb{R}_+,2}(f)\|_{L^1(\mathfrak{R}_+)} &\leq C\|R_{S_\lambda,2}(f)\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)}, \\ \|\mathcal{T}_{\mathbb{R}_+,1}\mathcal{T}_{\mathbb{R}_+,2}f\|_{L^1(\mathfrak{R}_+)} &\leq C\|R_{S_\lambda,1}R_{S_\lambda,2}(f)\|_{L^1(\mathfrak{R}_+)} + C\|R_{S_\lambda,1}(f)\|_{L^1(\mathfrak{R}_+)} \\ &\quad + C\|R_{S_\lambda,2}(f)\|_{L^1(\mathfrak{R}_+)} + C\|f\|_{L^1(\mathfrak{R}_+)}, \end{aligned}$$

implying that $f \in H_{\mathcal{T}}^1(\mathfrak{R}_+)$. □

7. APPLICATIONS: PROOF THEOREMS 1.9 AND 1.10

We first provide the definition of the classical product BMO space on $\mathbb{R}_+ \times \mathbb{R}_+$. We refer to the classical definition from Chang–Fefferman [CF]. See also the definition in product spaces of homogeneous type in [HLL1, HLL2]

Definition 7.1. Suppose $f \in L_{loc}^1(\mathfrak{R}_+)$. We say that $f \in \text{BMO}(\mathfrak{R}_+)$ if

$$\|f\|_{\text{BMO}(\mathfrak{R}_+)} := \sup_{\Omega} \frac{1}{|\Omega|} \sum_{R \subset \Omega} S_R^2(f) < \infty.$$

Here the suprema is taken over all open sets $\Omega \subset \mathbb{R}_+ \times \mathbb{R}_+$ with finite measures, the summation is taken over all dyadic rectangles $R \subset \Omega$, and

$$S_R^2(f) = \iint_{T(R)} |\psi_{t_1}^{(1)} \psi_{t_2}^{(2)} * f(y_1, y_2)|^2 \frac{dy_1 dt_1 dy_2 dt_2}{t_1 t_2}$$

with $\psi_{t_i}^{(i)}(y_i) := \frac{1}{t_i} \psi^{(1)}(y_i/t_i)$ where $\psi^{(i)} \in C^\infty(\mathbb{R}_+)$ supported in $(1/2, 2)$ for $i = 1, 2$, and $T(R) = I \times [\frac{|I|}{2}, |I|] \times J \times [\frac{|J|}{2}, |J|]$.

From [CF, THEOREM in Section II] (we also refer to [HLL1, Theorem 1.2]), we obtain that the duality of $H^1(\mathfrak{R}_+)$ is $\text{BMO}(\mathfrak{R}_+)$.

We now provide the definition of product BMO space associated with S_λ .

Definition 7.2. Suppose $f \in L_{loc}^1(\mathfrak{R}_+)$. We say that $f \in \text{BMO}_{S_\lambda}(\mathfrak{R}_+)$ if

$$\|f\|_{\text{BMO}_{S_\lambda}(\mathfrak{R}_+)} := \sup_{\Omega} \frac{1}{|\Omega|} \sum_{R \subset \Omega} S_R^2(f) < \infty.$$

Here the suprema is taken over all open sets $\Omega \subset \mathbb{R}_+ \times \mathbb{R}_+$ with finite measures, the summation is taken over all dyadic rectangles $R \subset \Omega$, and

$$S_R^2(f) = \iint_{T(R)} |Q_{t_1}^{(1)} Q_{t_2}^{(2)}(f)(y_1, y_2)|^2 \frac{dy_1 dt_1 dy_2 dt_2}{t_1 t_2}$$

with $Q_{t_i}^{(i)} := -t_i \frac{d}{dt_i} \mathbb{P}_{t_i}^{[\lambda]}$ for $i = 1, 2$.

From [DSTY, Theorem 4.4], we obtain that the duality of $H_{S_{Iz}}^1(\mathfrak{R}_+)$ is $\text{BMO}_{S_\lambda}(\mathfrak{R}_+)$.

Proof of Theorem 1.9. Suppose $f \in H^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$. Then we have the atomic decomposition of f (see [HLLin]):

$$f = \sum \lambda_j a_j$$

such that $\sum_{j=0}^\infty |\lambda_j| \leq 2\|f\|_{H^1(\mathfrak{R}_+)}$, where the series converges in the sense of $L^2(\mathfrak{R}_+)$ and $H^1(\mathfrak{R}_+)$, and each a_j is a product atom as follows.

A function $a(x_1, x_2) \in L^2(\mathfrak{R}_+)$ is a product atom if it satisfies

- 1) $\text{supp } a \subset \Omega$, where Ω is an open set of \mathfrak{R}_+ with finite measure;
- 2) $\|a\|_{L^2(\mathfrak{R}_+)} \leq |\Omega|^{-\frac{1}{2}}$;
- 3) a can be further decomposed into

$$a = \sum_{R \in m(\Omega)} a_R$$

where $m(\Omega)$ is the set of all maximal dyadic subrectangles of Ω , such that

- (i) $\text{supp } a_R \subset 10R$;
- (ii)

$$\int_{\mathbb{R}_+} a_R(x_1, x_2) dx_1 = \int_{\mathbb{R}_+} a_R(x_1, x_2) dx_2 = 0;$$

- (iii)

$$\sum_{R \in m(\Omega)} \|a_R\|_{L^2(\mathfrak{R}_+)}^2 \leq |\Omega|^{-1}.$$

As a consequence, it is direct that there exists a positive constant C such that for every product atom a ,

$$\begin{aligned} \|R_{S_\lambda, 1} R_{S_\lambda, 2}(a)\|_{L^1(\mathfrak{R}_+)} &\leq C, \\ \|R_{S_\lambda, 1}(a)\|_{L^1(\mathfrak{R}_+)} &\leq C, \end{aligned}$$

and

$$\|R_{S_\lambda, 2}(a)\|_{L^1(\mathfrak{R}_+)} \leq C,$$

all implying

$$\|a\|_{H_{Riesz}^1(\mathfrak{R}_+)} \leq C.$$

For the detail of the proof, we refer to [HLLin]. Thus, for $f \in H^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$, we have

$$\|f\|_{H_{Riesz}^1(\mathfrak{R}_+)} \leq \sum_{j=0}^{\infty} |\lambda_j| \|a\|_{H_{Riesz}^1(\mathfrak{R}_+)} \leq C \|f\|_{H^1(\mathfrak{R}_+)}.$$

Since $H^1(\mathfrak{R}_+) \cap L^2(\mathfrak{R}_+)$ is dense in $H^1(\mathfrak{R}_+)$, we have that for every $f \in H^1(\mathfrak{R}_+)$,

$$\|f\|_{H_{Riesz}^1(\mathfrak{R}_+)} \leq C \|f\|_{H^1(\mathfrak{R}_+)}.$$

Thus, we get that the classical product Hardy space $H^1(\mathfrak{R}_+)$ is a subspace of $H_{S_\lambda}^1(\mathfrak{R}_+)$, i.e.

$$H^1(\mathfrak{R}_+) \subset H_{S_\lambda}^1(\mathfrak{R}_+).$$

Next, we point out that $H^1(\mathfrak{R}_+)$ is a proper subspace of $H_{S_\lambda}^1(\mathfrak{R}_+)$. To see this, note that from Theorem 1.8, we obtain that $H_{S_\lambda}^1(\mathfrak{R}_+)$ coincides with $H_o^1(\mathfrak{R}_+)$. We now choose

$$f(x_1, x_2) = \chi_{Q_0}(x_1, x_2),$$

where $Q_0 = (0, 1] \times (0, 1]$ is the unit cube in $\mathbb{R} \times \mathbb{R}$. It is direct to see that the product odd extension f_o is in $H^1(\mathbb{R} \times \mathbb{R})$, and hence this function f is in $H_{S_\lambda}^1(\mathfrak{R}_+)$. However, it is not in the product Hardy space $H^1(\mathfrak{R}_+)$ since it lacks cancellation. Thus, we further have

$$H^1(\mathfrak{R}_+) \subsetneq H_{S_\lambda}^1(\mathfrak{R}_+).$$

As a consequence, we obtain that $\text{BMO}_{S_\lambda}(\mathfrak{R}_+)$ is contained in the classical product BMO space $\text{BMO}(\mathfrak{R}_+)$, i.e.,

$$\text{BMO}_{S_\lambda}(\mathfrak{R}_+) \subsetneq \text{BMO}(\mathfrak{R}_+).$$

□

We now provide the proof of Theorem 1.10.

Proof of Theorem 1.10. From the kernel estimates of $(i)'$ and $(ii)'$ of the Riesz transform as in Section 2.4, we see that $R_{S_\lambda,1}$ and $R_{S_\lambda,2}$ are standard Calderón–Zygmund operators. Hence, the composition $R_{S_\lambda,1}R_{S_\lambda,2}$ are the standard product Calderón–Zygmund operators.

Based on the general result of upper bound for the iterated commutator and product BMO space on space of homogeneous type ([DLOWY, Theorem 3.3]), we obtain that

$$\|[[b, R_{S_\lambda,1}], R_{S_\lambda,2}]\|_{L^2(\mathfrak{R}_+) \rightarrow L^2(\mathfrak{R}_+)} \lesssim \|b\|_{\text{BMO}_{S_\lambda}(\mathfrak{R}_+)},$$

In fact, for functions b in the classical product BMO space $\text{BMO}(\mathfrak{R}_+)$, we also have

$$\|[[b, R_{S_\lambda,1}], R_{S_\lambda,2}]\|_{L^2(\mathfrak{R}_+) \rightarrow L^2(\mathfrak{R}_+)} \lesssim \|b\|_{\text{BMO}(\mathfrak{R}_+)}.$$

From Theorem 1.9, we know that

$$\text{BMO}_{S_\lambda}(\mathfrak{R}_+) \subsetneq \text{BMO}(\mathfrak{R}_+).$$

We now choose $b_0 \in \text{BMO}(\mathfrak{R}_+) \setminus \text{BMO}_{S_\lambda}(\mathfrak{R}_+)$, then we know that the iterated commutator $[[b_0, R_{S_\lambda,1}], R_{S_\lambda,2}]$ is bounded, which gives

$$\infty = \|b\|_{\text{BMO}_{S_\lambda}(\mathfrak{R}_+)} \lesssim \|[[b_0, R_{S_\lambda,1}], R_{S_\lambda,2}]\|_{L^2(\mathfrak{R}_+) \rightarrow L^2(\mathfrak{R}_+)} < \infty.$$

□

Acknowledgments: J. Betancor is supported by MTM2016-79436-P. X. T. Duong and J. Li are supported by ARC DP 160100153. J. Li is also supported by a Macquarie University New Staff Grant. B. D. Wick's research supported in part by National Science Foundation DMS grant #1560955. D. Yang is supported by the NNSF of China (Grant No. 11571289). Ji Li would like to thank Yumeng Ou for helpful discussions.

REFERENCES

- [AM] K. F. Andersen and B. Muckenhoupt, Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions, *Studia Math.*, **72** (1982), 9–26.
- [AK] K. F. Andersen and R. A. Kerman, Weighted norm inequalities for generalized Hankel conjugate transformations, *Studia Math.*, **71** (1981/82), 15–26.
- [BCFR] J. J. Betancor, A. Chicco Ruiz, J. C. Fariña and L. Rodríguez-Mesa, Maximal operators, Riesz transforms and Littlewood-Paley functions associated with Bessel operators on BMO, *J. Math. Anal. Appl.*, **363** (2010), 310–326.
- [BCFR2] J. J. Betancor, A. Chicco Ruiz, J. C. Fariña and L. Rodríguez-Mesa, Solutions of Weinstein equations representable by Bessel Poisson integrals of BMO functions, *J. Math. Anal. Appl.* **431** (2015), 440–470.
- [BFBMT] J. J. Betancor, J. C. Fariña, D. Buraczewski, T. Martínez and J. L. Torrea, Riesz transform related to Bessel operators, *Proc. Roy. Soc. Edinburgh Sect. A*, **137** (2007), 701–725.
- [BFS] J. J. Betancor, J. C. Fariña and A. Sanabria, On Littlewood-Paley functions associated with Bessel operators, *Glasg. Math. J.*, **51** (2009), 55–70.
- [BDT] J. J. Betancor, J. Dziubański and J. L. Torrea, On Hardy spaces associated with Bessel operators, *J. Anal. Math.*, **107** (2009) 195–219.
- [BHNv] J. J. Betancor, E. Harboure, A. Nowak and B. Viviani, Mapping properties of functional operators in harmonic analysis related to Bessel operators, *Studia Math.*, **197** (2010), 101–140.
- [CF] S.-Y. A. Chang and R. Fefferman, A continuous version of the duality of H^1 with BMO on the bi-disc, *Ann. Math.*, **112** (1980), 179–201.
- [CDLWY] P. Chen, X. Duong, J. Li, L. Ward and L. X. Yan, Product Hardy spaces associated to operators with heat kernel bounds on spaces of homogeneous type, *Math. Z.*, **282** (2016), 1033–1065.
- [CW] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.*, **83** (1977), 569–645.
- [CRW] R. R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. of Math.*, **103** (1976), 611–635.
- [CS] T. Coulhon and A. Sikora, Gaussian heat kernel upper bounds via Phragmén-Lindelöf theorem, *Proc. Lond. Math.*, **96** (2008), 507–544.
- [DO] L. Dalenc and Y. Ou, Upper Bound for Multi-parameter Iterated Commutators, *Publ. Mat.*, **60** (2016), 191–220.
- [DSTY] D. G. Deng, L. Song, C. Q. Tan and L. X. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds on product domains, *J. Geom. Anal.*, **17** (2007), 455–483.
- [DLOWY] X. T. Duong, J. Li, Y. Ou, B. D. Wick and D. Yang, Product BMO, little BMO and Riesz Commutators in the Bessel setting, arXiv:1609.01368.
- [DLWY] X. T. Duong, Ji Li, B. D. Wick and D. Yang, Hardy space via factorization, and BMO space via commutators in the Bessel setting, *Indiana Univ. Math. J.*, to appear.
- [DLWY2] X. T. Duong, Ji Li, B. D. Wick and D. Yang, Characterizations of product Hardy spaces in Bessel setting, arXiv:1606.03500.
- [DLY] X. T. Duong, J. Li and L. X. Yan, Endpoint estimates for singular integrals with non-smooth kernels on product spaces, arXiv:1509.07548.
- [DY1] X. T. Duong and L. X. Yan, New function spaces of BMO type, John-Nirenberg inequality, interpolation and applications, *Comm. Pure Appl. Math.*, **58** (2005), 1375–1420.
- [DY2] X. T. Duong and L. X. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds, *J. Amer. Math. Soc.*, **18** (2005), 943–973.
- [EMOT] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Tables of Integral Transforms. Vol. II. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.
- [FSt] R. Fefferman and E. M. Stein, Singular integrals on product spaces, *Adv. Math.*, **45** (1982), 117–143.
- [FL] S. H. Ferguson and M. T. Lacey, A characterization of product BMO by commutators, *Acta Math.*, **189** (2002), 143–160.
- [FS] S. H. Ferguson and C. Sadosky, Characterizations of bounded mean oscillation on the polydisk in terms of Hankel operators and Carleson measures, *J. Anal. Math.*, **81** (2000), 239–267.
- [F] S. Fridli, Hardy spaces generated by an integrability condition, *Journal of Approximation Theory*, **113** (2001), 91–109.
- [GS] R. Gundy and E. M. Stein, H^p theory for the poly-disc, *Proc. Nat. Acad. Sci. U.S.A.*, **76** (1979), 1026–1029.
- [H] D. T. Haimo, Integral equations associated with Hankel convolutions, *Trans. Amer. Math. Soc.*, **116** (1965), 330–375.
- [HLLin] Y. Han, J. Li, and C. C. Lin, Criteria of the L^2 boundedness and sharp endpoint estimates for singular integral operators on product spaces of homogeneous type, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, to appear.

- [HLL1] Y. Han, J. Li and G. Lu, Duality of multiparameter Hardy spaces H^p on spaces of homogeneous type, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **9** (2010), 645–685.
- [HLL2] Y. Han, J. Li and G. Lu, Multiparameter Hardy space theory on Carnot-Carathéodory spaces and product spaces of homogeneous type, *Trans. Amer. Math. Soc.*, **365** (2013), 319–360.
- [HLM MY] S. Hofmann, G. Z. Lu, D. Mitrea, M. Mitrea and L. X. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates, *Mem. Amer. Math. Soc.*, **214** (2011), no. 1007, vi+78 pp.
- [J1] J.-L. Journé, Calderón-Zygmund operators on product spaces, *Rev. Mat. Iberoam.*, **1** (1985), 55–91.
- [J2] J.-L. Journé, A covering lemma for product spaces, *Proc. Amer. Math. Soc.*, **96** (1986), 593–598.
- [L] M. Lacey, Lectures on Nehari’s theorem on the polydisk, Topics in harmonic analysis and ergodic theory. *Contemp. Math.*, **444** (2007), 185–213.
- [LPPW] M. Lacey, S. Petermichl, J. Pipher and B. D. Wick, Multiparameter Riesz commutators, *Amer. J. Math.*, **131** (2009), 731–769.
- [LPPW2] M. Lacey, S. Petermichl, J. Pipher and B. D. Wick, Iterated Riesz commutators: a simple proof of boundedness, Harmonic analysis and partial differential equations, *Contemp. Math.*, **505** (2010), 171–178.
- [LT] M. Lacey and E. Terwilleger, Hankel operators in several complex variables and product BMO, *Houston J. Math.*, **35** (2009), 159–183.
- [Le] N. N. Lebedev, Special functions and their applications. Selected Russian Publications in the Mathematical Sciences. Prentice-Hall, INC., Englewood Cliffs, N.J., 1965.
- [LS] S. Liu and L. Song, The atomic decomposition of weighted Hardy spaces associated to self-adjoint operators in product spaces, *J. Math. Anal. Appl.*, **443** (2016), 92–115.
- [M] K.G. Merryfield, On the area integral, Carleson measures and H^p in the polydisc, *Indiana Univ. Math. J.*, **34** (1985), 663–685.
- [MSt] B. Muckenhoupt and E. M. Stein, Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.*, **118** (1965), 17–92.
- [P] J. Pipher, Journé’s covering lemma and its extension to higher dimensions, *Duke Math. J.*, **53** (1986), 683–690.
- [Si] A. Sikora, Riesz transform, Gaussian bounds and the method of wave equation, *Math. Z.* **247** (2004), 643–662.
- [St93] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, NJ, 1993.
- [St70] E. M. Stein, Topics in Harmonic Analysis Related to the Littlewood-Paley Theory, Princeton Univ. Press, 1970.
- [SW] E. M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean spaces. Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J., 1971.
- [STY] L. Song, C. Q. Tan and L. Yan, An atomic decomposition for Hardy spaces associated to Schrödinger operators, *J. Aust. Math. Soc.*, **91**, 125–144.
- [V] M. Villani, Riesz transforms associated to Bessel operators, *Illinois J. Math.*, **52** (2008), 77–89.
- [YY] Da. Yang and Do. Yang, Real-variable characterizations of Hardy spaces associated with Bessel operators, *Anal. Appl. (Singap.)*, **9** (2011), 345–368.

JORGE J. BETANCOR, DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA, CAMPUS DE ANCHIETA, AVDA. ASTROFÍSICO FRANCISCO SÁNCHEZ, s/n, 38271, LA LAGUNA (STA. CRUZ DE TENERIFE), SPAIN.

E-mail address: jbetanco@ull.es

XUAN THINH DUONG, DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NSW 2019, AUSTRALIA

E-mail address: xuan.duong@mq.edu.au

Ji LI, DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NSW 2019, AUSTRALIA

E-mail address: ji.li@mq.edu.au

BRETT D. WICK, DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY – ST. LOUIS, ST. LOUIS, MO 63130-4899 USA

E-mail address: wick@math.wustl.edu

DONGYONG YANG, SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, XIAMEN 361005, CHINA

E-mail address: dyyang@xmu.edu.cn